Efficient Algorithms for Folding Problem of Regular Polyhedra

Ryuhei UEHARA, JAIST, Japan (uehara@jaist.ac.jp)

We investigate the folding problem when both a polygon and a polyhedron are explicitly given; for a given polygon P and a polyhedron Q, it asks if P can fold to Q or not. This is a natural problem, however, there are few results so far. When Q is a regular tetrahedron, we have a mathematical characterization of its net [1]; according to this result, P can fold to Q if and only if P is a kind of tiling. The folding problem has been investigated when Q is a box. Some special cases were investigated [4, 7] and it is solved in the general case [6].

We show that the folding problem can be solved efficiently for a regular polyhedron (a. k. a. Platonic solid). aka While there are five Platonic solids, our main result consists of four algorithms. That is, we investigate four problems depending on the target solid, and show pseudo-polynomial time algorithms for them.

The first algorithm solves the folding problem for a regular tetrahedron. We give a bit stronger algorithm that solves the folding problem for a tetramonohedron, which is a tetrahedron consists of four congruent acute triangles. As already mentioned, a mathematical characterization of a net of a regular tetrahedron and its extension to a tetramonohedron is also given by Akiyama and Nara [1, 3]. (The details can be found in a book [2, Chapter 3]). However, as far as the authors know, any *algorithm* for checking these characterizations has never been given explicitly (we thank an anonymous referee of [5], who mentioned this point). We give a pseudo-polynomial time algorithm for the folding problem for a tetramonohedron. The next algorithm solves the folding problem for a regular cube. As mentioned above, there is a known algorithm for this problem [6]. We improve this running time from $O(D^7n^2(D^5+\log n))$ time to $O(D^2n^3)$ time.

The third algorithm solves the folding problem for a regular dodecahedron. The last algorithm solves the folding problem for a set of special convex deltahedra. Usually, a *deltahedron* means a polyhedron whose faces are congruent equilateral triangles. In our algorithm, it cannot deal with a vertex of curvature 180°, which consists of three equilateral triangles. Therefore, this algorithm cannot deal with a regular tetrahedron. On the other hand, it can deal with a vertex of curvature 360°, where six equilateral triangles make a flat hexagonal face. That is, we allow each face to consist of coplanar regular triangles, or each face can be any convex *polyiamond*. Thus there are an infinite of non-strictly convex deltahedra that our algorithm can deal with. In summary, if Q has a non-strictly convex deltahedra with no vertex of curvature 180°, our algorithm solves the folding problem in pseudo-polynomial time.

References

[1] Akiyama, J.: Tile-Makers and Semi-Tile-Makers. The Mathematical Association of America **Monthly 114**, 602–609 (2007)

[2] Akiyama, J., Matsunaga, K.: Treks into Intuitive Geometry. Springer (2015)

[3] Akiyama, J., Nara, C.: Developments of Polyhedra Using Oblique Coordinates. J. Indonesia. Math. Soc. **13**(1), 99–114 (2007)

[4] Horiyama, T., Mizunashi, K.: Folding Orthogonal Polygons into Rectangular Boxes. In: In Proc. of 19th Japan-Korea Joint Workshop on Algorithms and Computation (WAAC 2016) (2016)

[5] Kamata, T., Kadoguchi, A., Horiyama, T., Uehara, R.: Efficient Folding Algorithms for Regular Polyhedra. In: 32nd Canadian Conference on Computational Geometry (CCCG 2020), pp. 131–137 (2020)

[6] Mizunashi, K., Horiyama, T., Uehara, R.: Efficient Algorithm for Box Folding. Journal of Graph Algorithms and Applications **24**(2), 89–103 (2020)

[7] Xu, D., Horiyama, T., Shirakawa, T., Uehara, R.: Common Developments of Three Incongruent Boxes of Area 30. Computational Geometry: Theory and Applications **64**, 1–17 (2017). DOI:10.1016/j.comgeo.2017.03.001