A thresholding algorithm for hyperbolic mean curvature flow

Elliott Ginder

北海道大学 電子科学研究所

Joint work with

Karel Svadlenka

金沢大学　理工研究域

反応拡散現象にみられる境界層とその周辺の数理

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Outline

1. Curvature dependent motions
   - $\star$ $v = -\kappa$ (Mean curvature flow)
   - $\star$ $\frac{dv}{dt} = -\kappa$ (Hyperbolic MCF)

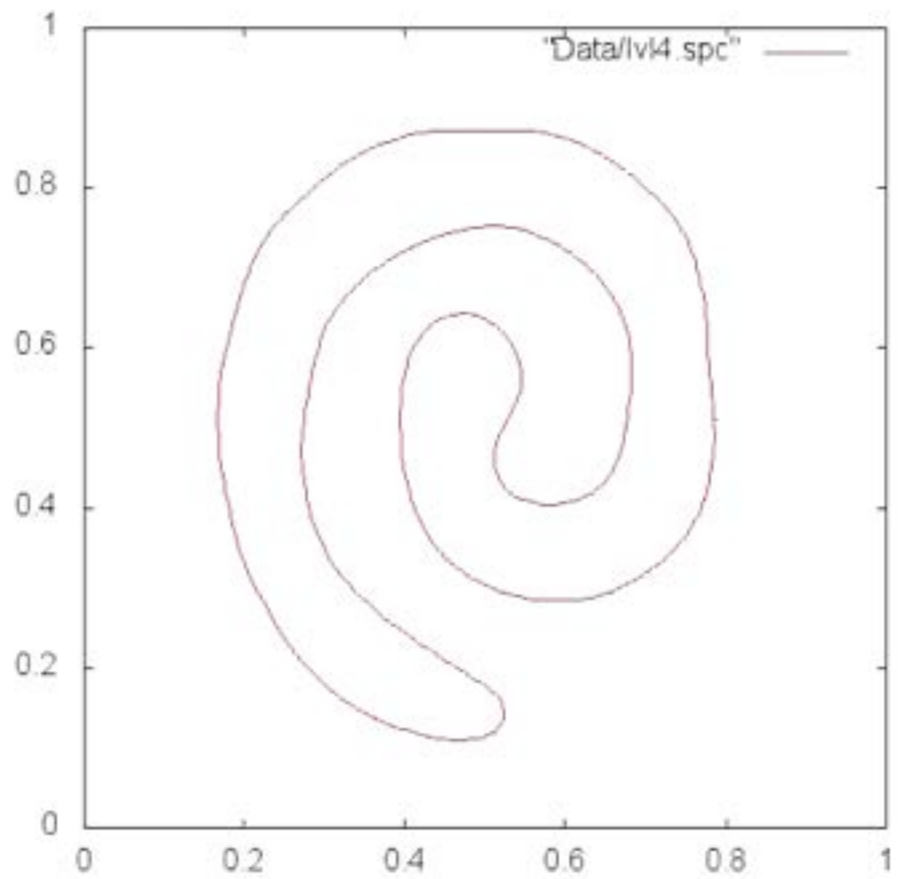
   (see J. Math. Pures Appl., P. LeFloch and K. Smoczyk 2008)

2. An approximation method for HMCF
   - $\star$ The original BMO (mean curvature flow)
   - $\star$ The hyperbolic BMO

     $+$ Justification of our thresholding algorithm.

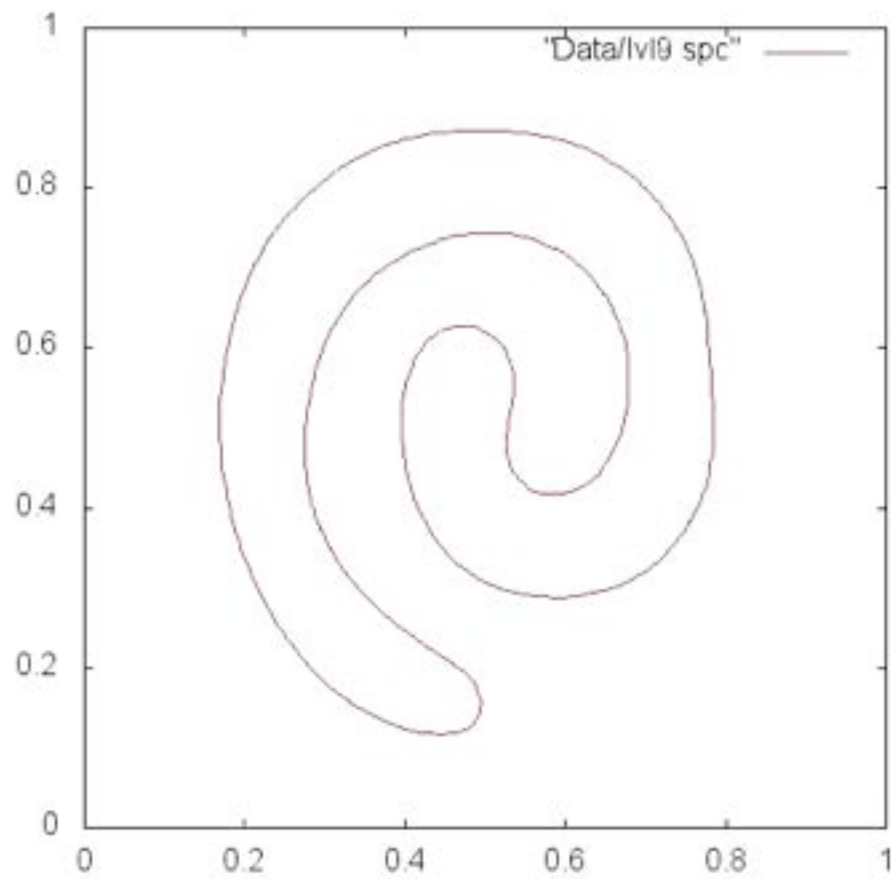
3. Application of the numerical method.
   - $\star$ Hyperbolic mean curvature flow (HMCF)
   - $\star$ Multiphase HMCF
   - $\star$ Volume preserving HMCF, contact angles, numerical tests

What does curvature-driven acceleration look like?



Mean Curvature Flow

$$v = -\kappa$$

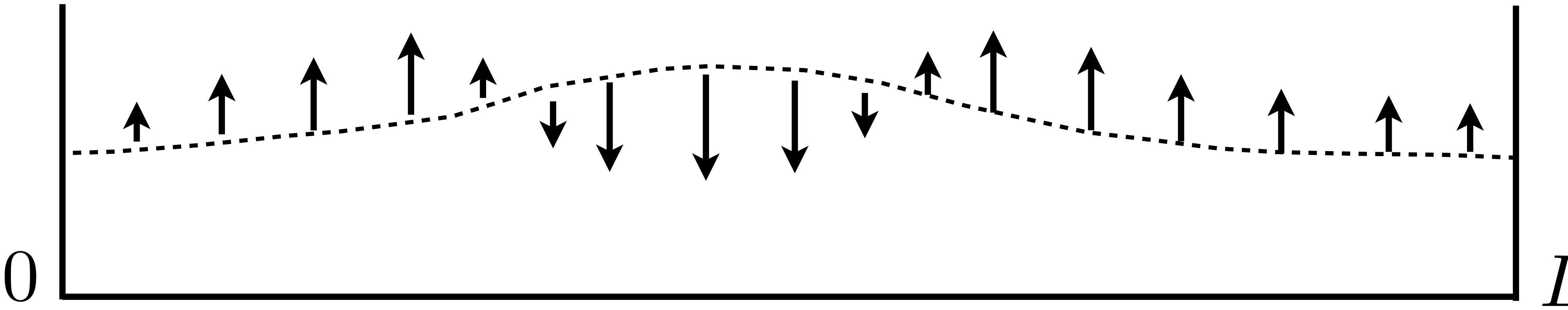


Hyperbolic Mean Curvature Flow

$$\frac{dv}{dt} = -\kappa$$

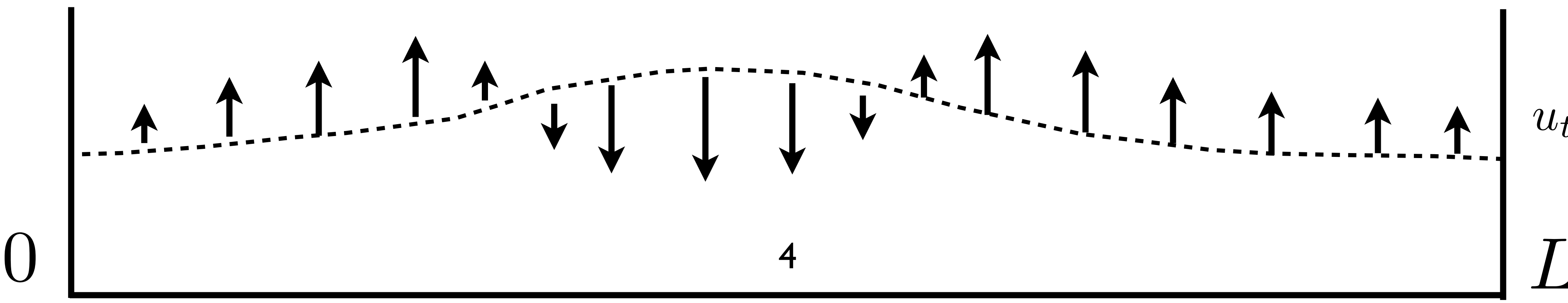
Physical interpretation (membrane motion)

Wave equation



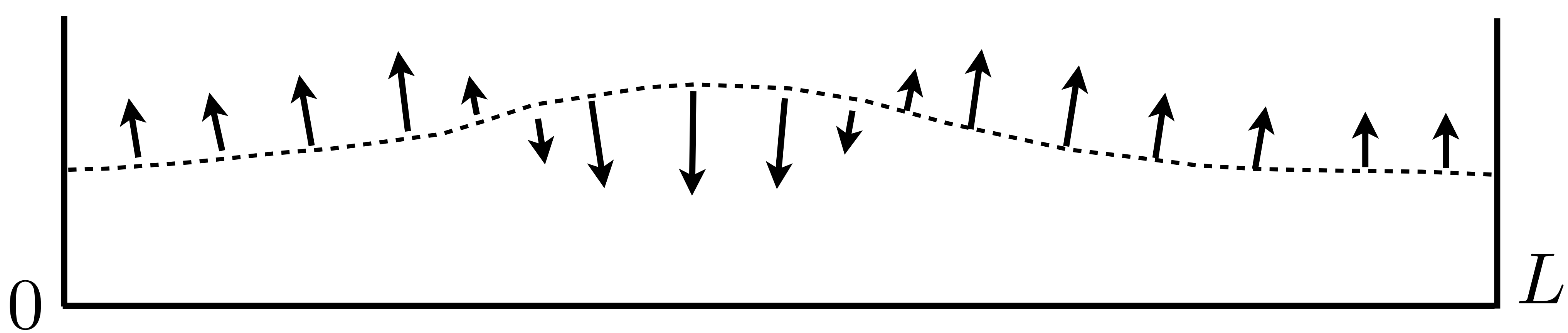
$$u_{tt} = u_{xx}$$

Nonlinear wave equation



$$u_{tt} = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

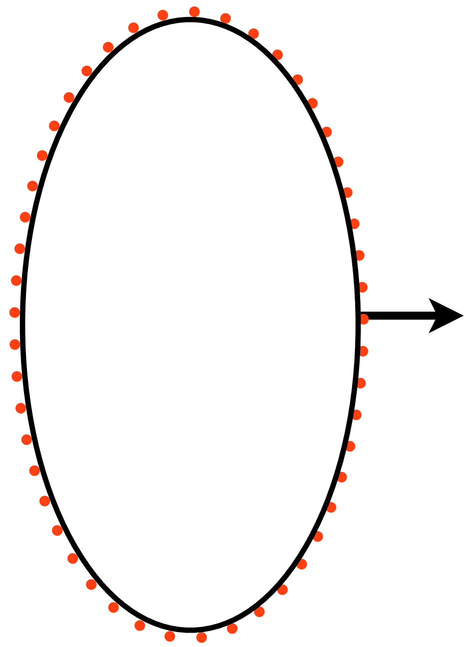
Hyperbolic mean curvature flow



$$\frac{dv}{dt} = -\kappa$$

Computational Tools

Front tracking



$$\frac{dv}{dt} = -\kappa$$

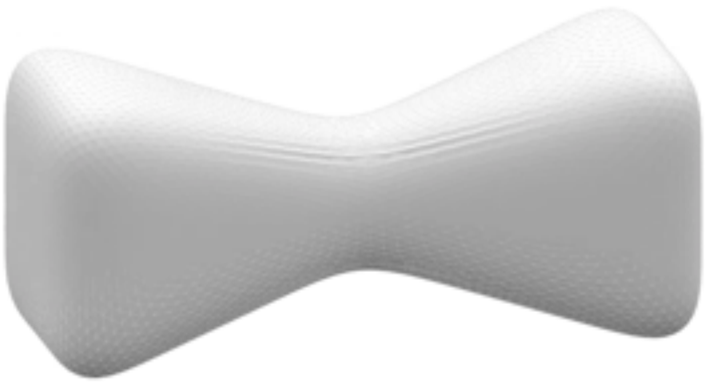
- explicitly evolve nodes
- simple to implement
- development of singularities causes difficulty

Threshold dynamical algorithms

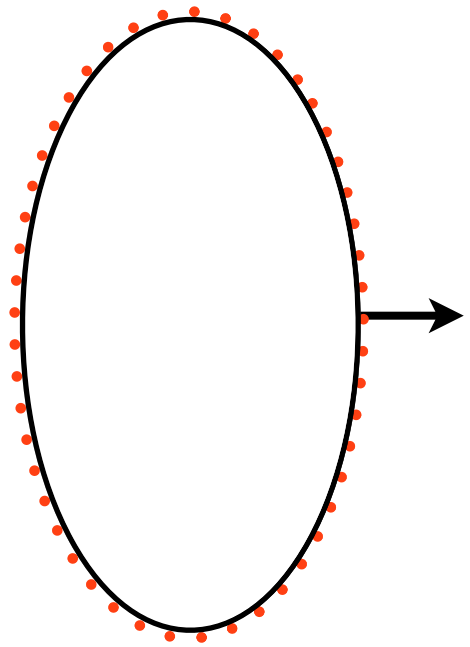
Multiphase Vol. Pres.

Merging

Topological changes



Front tracking



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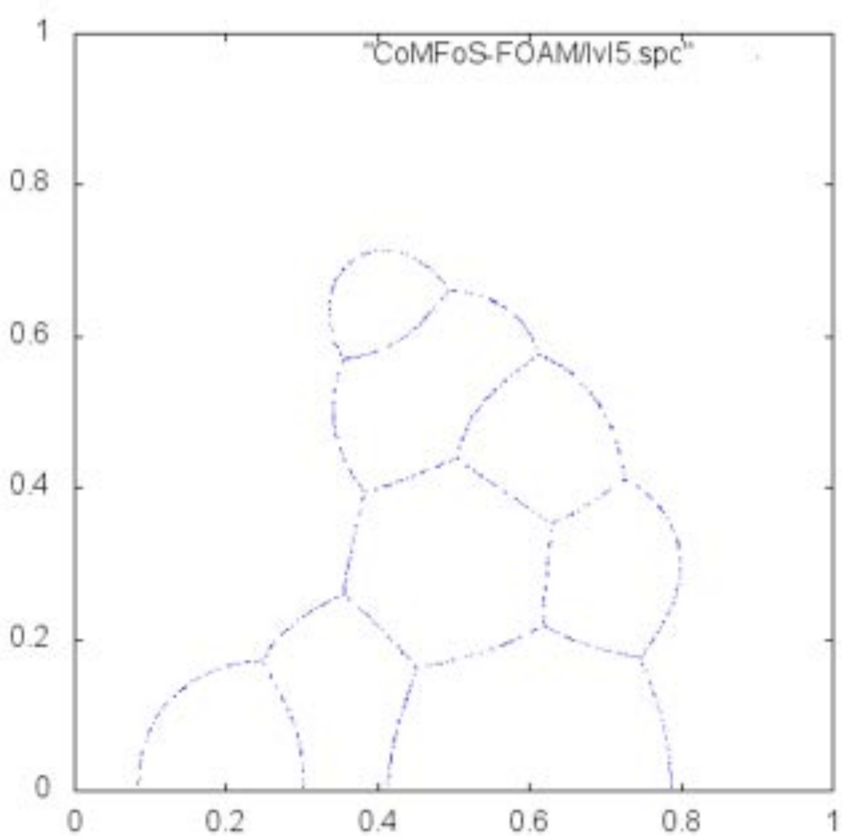
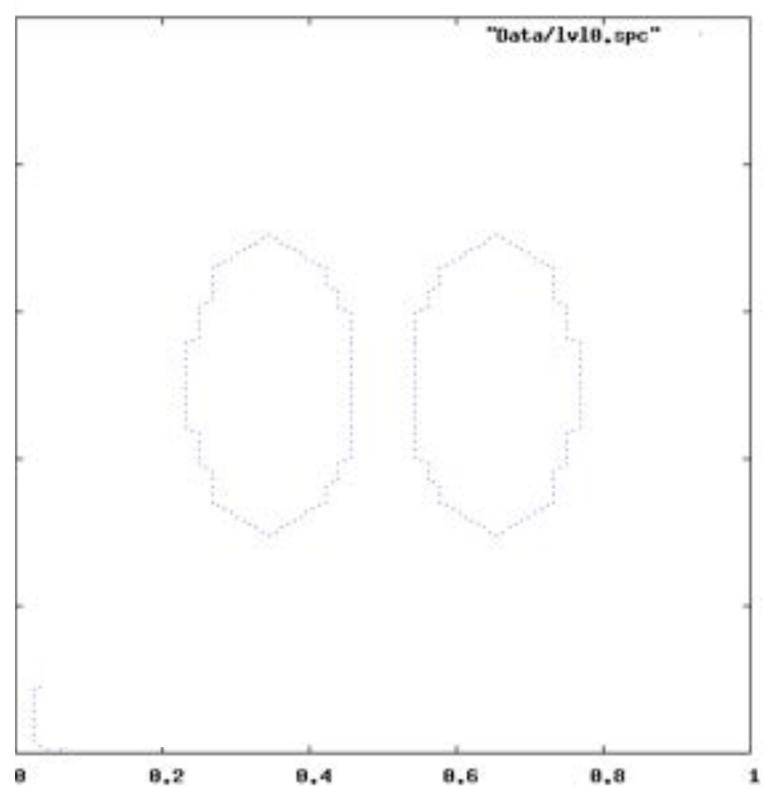
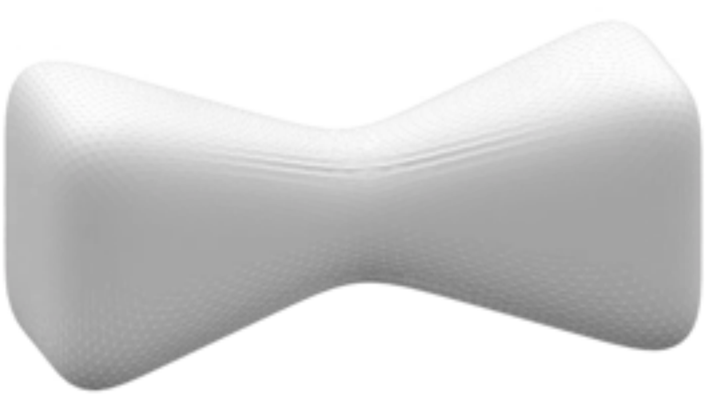
Multiphase Vol. Pres.



Merging

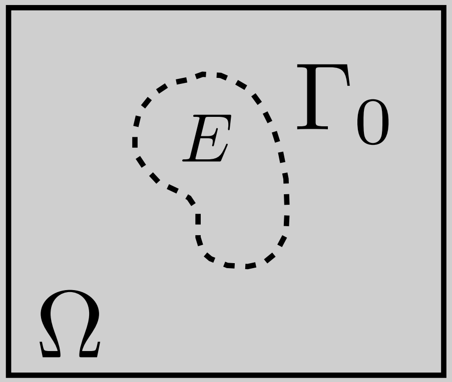


Topological changes

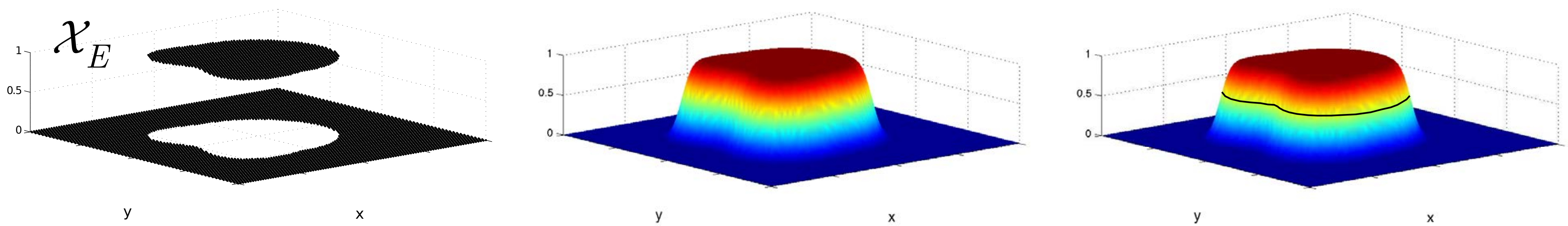


The scalar BMO algorithm (1992) (**B**ence-**M**erriman-**O**sher)

Evolve by mean curvature flow $\quad v = -\kappa \quad\quad 0 < \Delta t \ll 1$



$$\begin{cases} u_t = \Delta u & \text{in } (0, \Delta t) \times \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } (0, \Delta t) \times \partial\Omega \\ u(t = 0, x) = \chi_E & \text{in } \Omega \end{cases}$$



Initial condition $\longrightarrow$ Heat operator $\longrightarrow$ Truncate

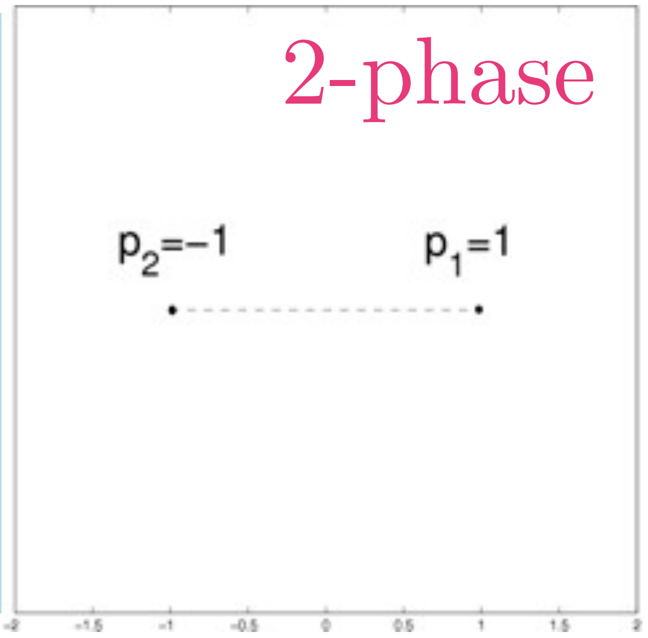
$\Gamma_k = \partial\{u(t = \Delta t, x) > 1/2\}$

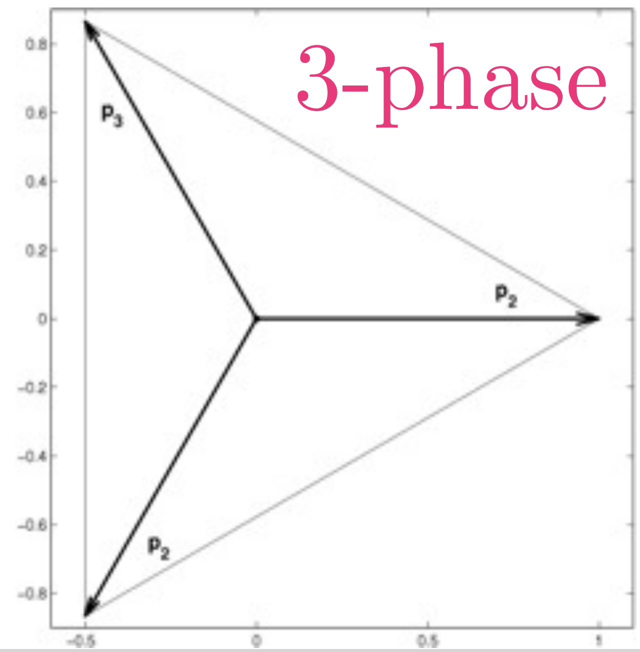
$u_0 = \chi_{\{u(t=\Delta t, x) > 1/2\}}$

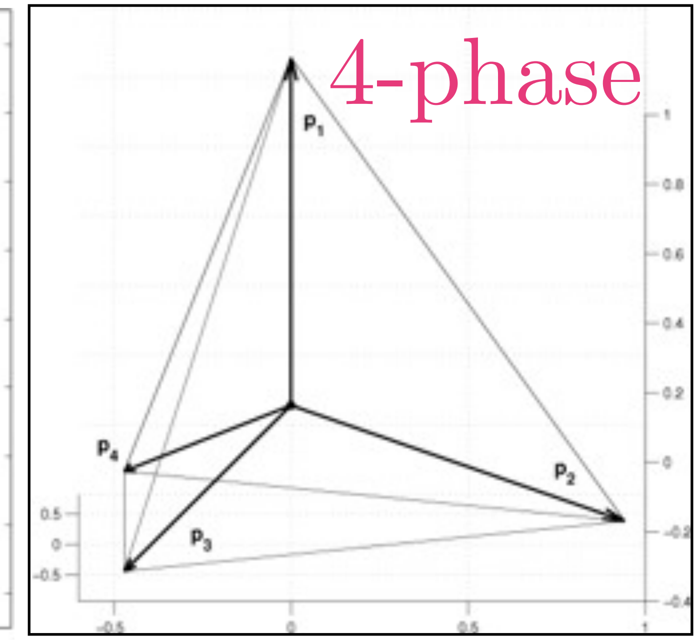
Reformulated multiphase (vector-type) BMO algorithm:

$\mathbf{p}_k \in \mathbf{R}^{N-1}$:

coordinates are given by the vertices of a *regular simplex* in $\mathbf{R}^N$.







V-BMO

for $k = 1, \ldots, MAX$ do $\qquad 0 < \Delta t \ll 1$

set $\mathbf{u}_0(x) = \mathbf{p}_i$ for $x \in E_i \quad i = 1, \ldots, N$ — Initialize

$$\begin{cases} \mathbf{u}_t = \Delta \mathbf{u} & \text{in } (0, \Delta t) \times \Omega \\ \dfrac{\partial \mathbf{u}}{\partial \nu} = 0 & \text{on } (0, \Delta t) \times \partial\Omega \\ \mathbf{u}(t = 0, x) = \mathbf{u}_0 & \text{in } \Omega \end{cases}$$

Solve the heat equation for a small time Δt

Truncation

$$E_i = \{x \in \Omega : \mathbf{u}(\Delta t, x) \cdot \mathbf{p}_i = \max_{j=1,\ldots,N} \mathbf{u}(\Delta t, x) \cdot \mathbf{p}_j\}, i = 1, \ldots, N$$

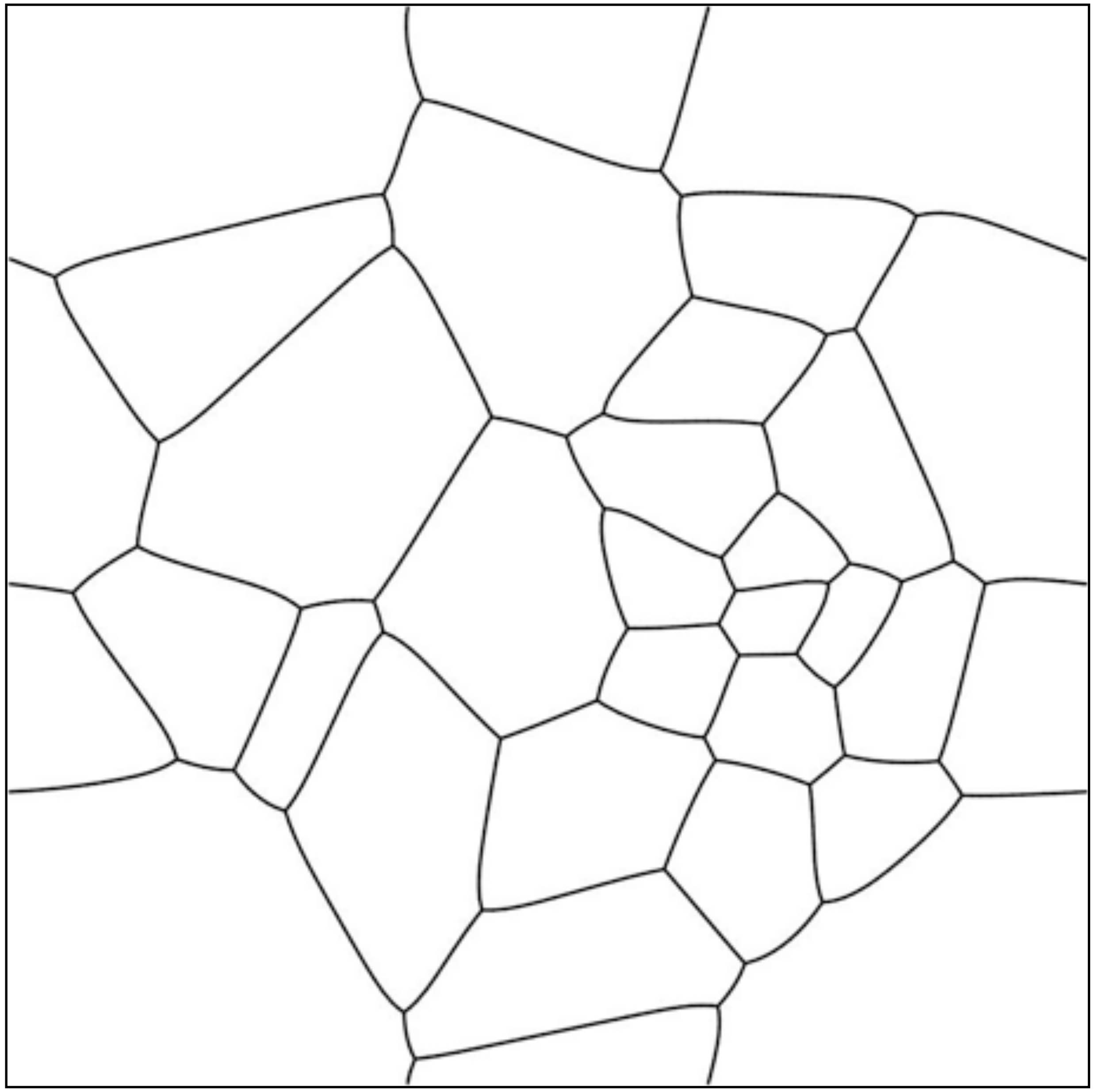
$$\Gamma_k^i = \partial E_i, i = 1, \ldots, N$$

Evolve the curve

end

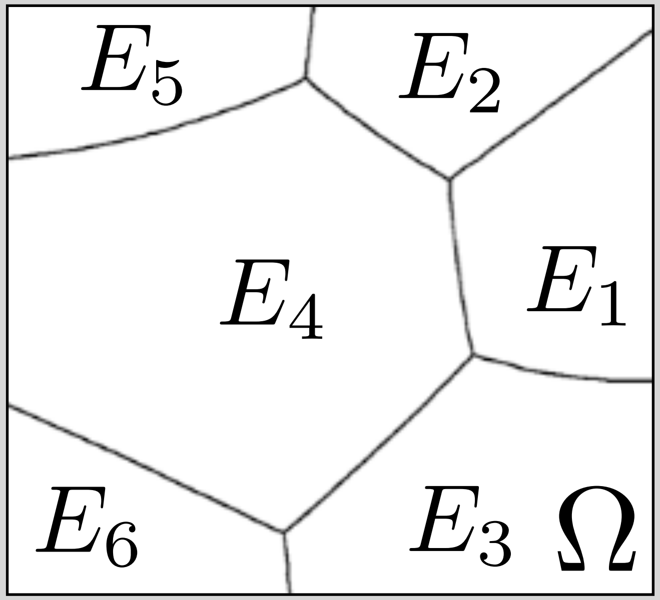
Curvature Dependent Motions

$$v = -\kappa$$



mean curvature flow

Volume constrained motions



PDE

$$\mathbf{u}_t = \Delta \mathbf{u} + \sum_{i=1}^{N-1} \lambda_i \mathbf{p}_i \mathcal{H}^{m-1}\lfloor_{\partial E_i}$$

Interfacial motion

$$v_i = -\kappa_i + \tilde{\lambda}_i$$

Vector-type minimizing movement: $\mathbf{u}, \mathbf{u}_{n-1} \in H^1(\Omega; \mathbf{R}^{N-1})$

$$\mathcal{F}_n(\mathbf{u}) = \int_\Omega \left(\frac{|\mathbf{u} - \mathbf{u}_{n-1}|^2}{2h} + \frac{|\nabla \mathbf{u}|^2}{2} \right) dx$$

$$+ \sum_{k=1}^{N-1} \frac{1}{\tilde{\epsilon}} \left| V_k - meas(E_k^n) \right|^2$$

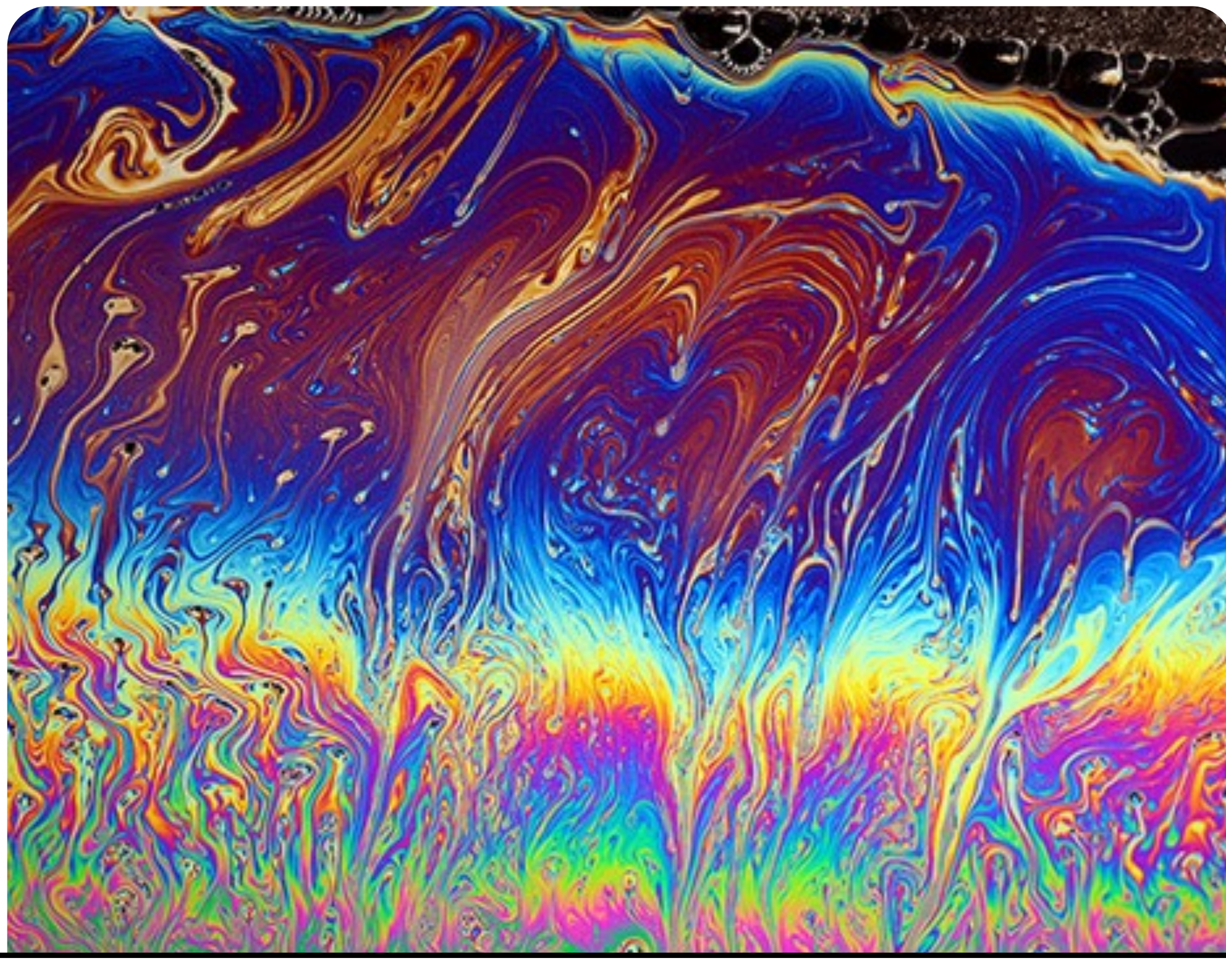
Heat Type

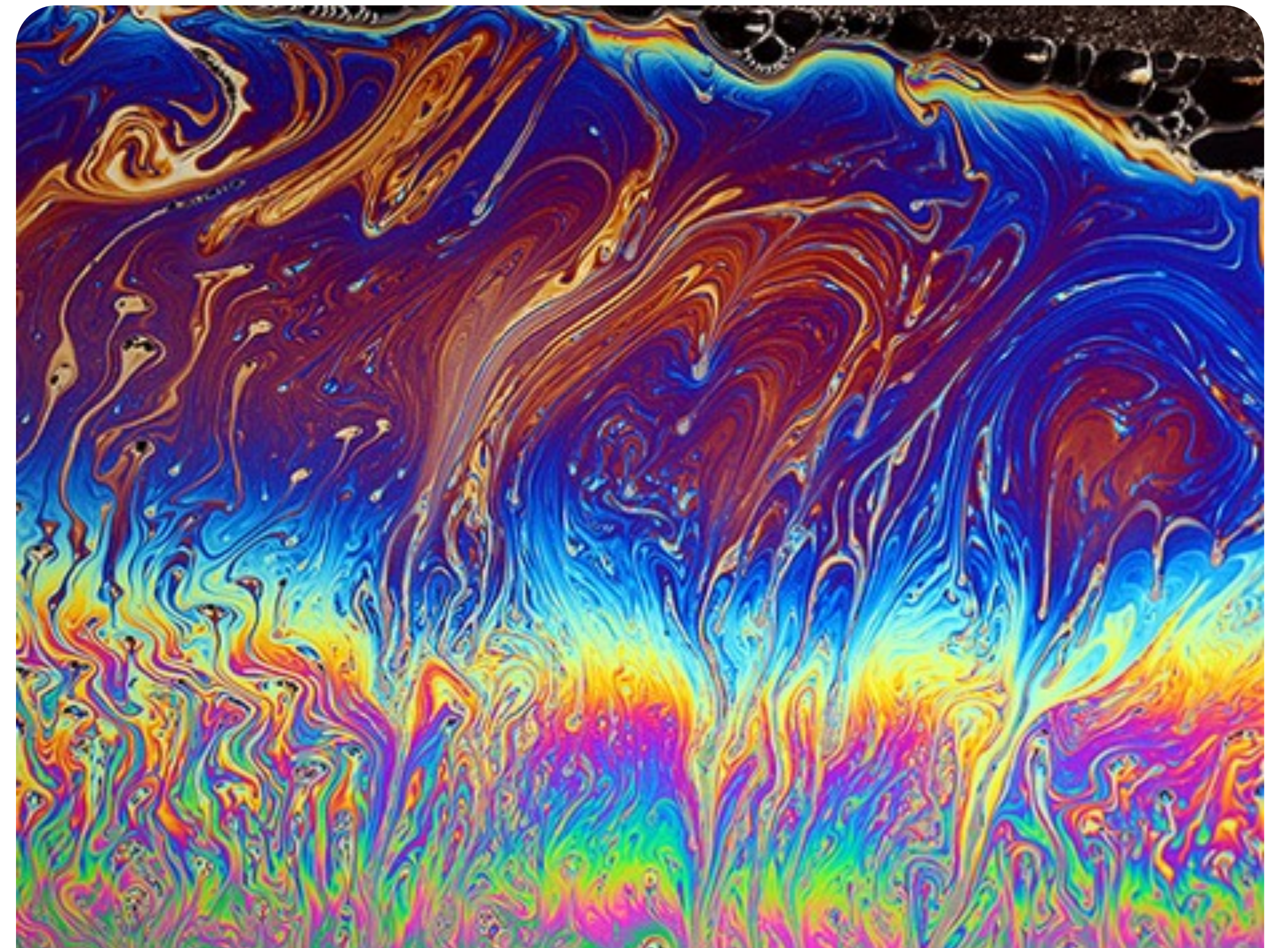
$\tilde{\epsilon} > 0$

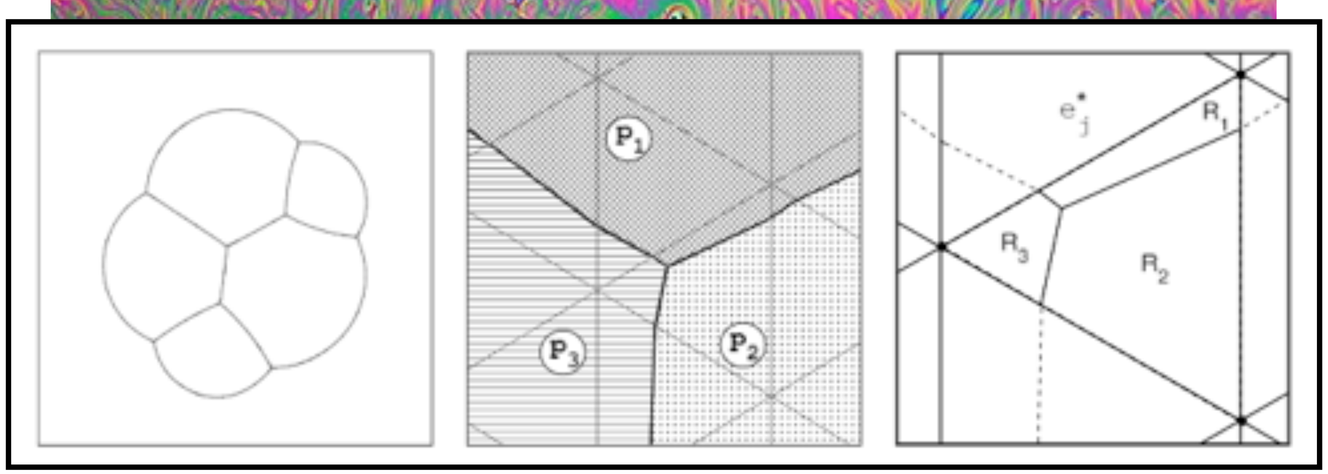
Penalties

$$E_k^n = \{ x \in \mathbf{R}^m : \mathbf{u}(x) \cdot \mathbf{p}_k = \max_{i=1,\dots,N} \mathbf{u}(x) \cdot \mathbf{p}_i \}$$

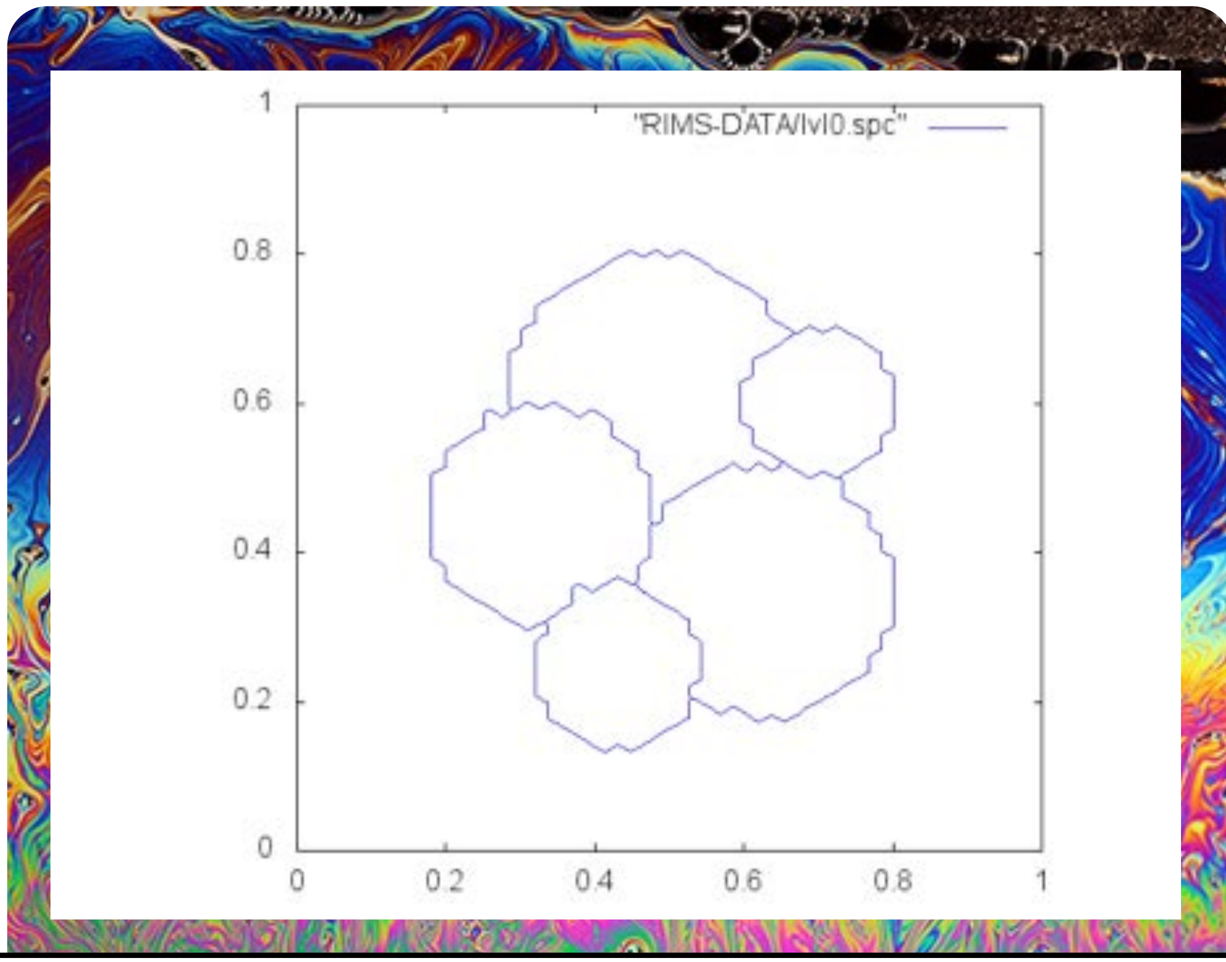
Multi-phase volume preserving

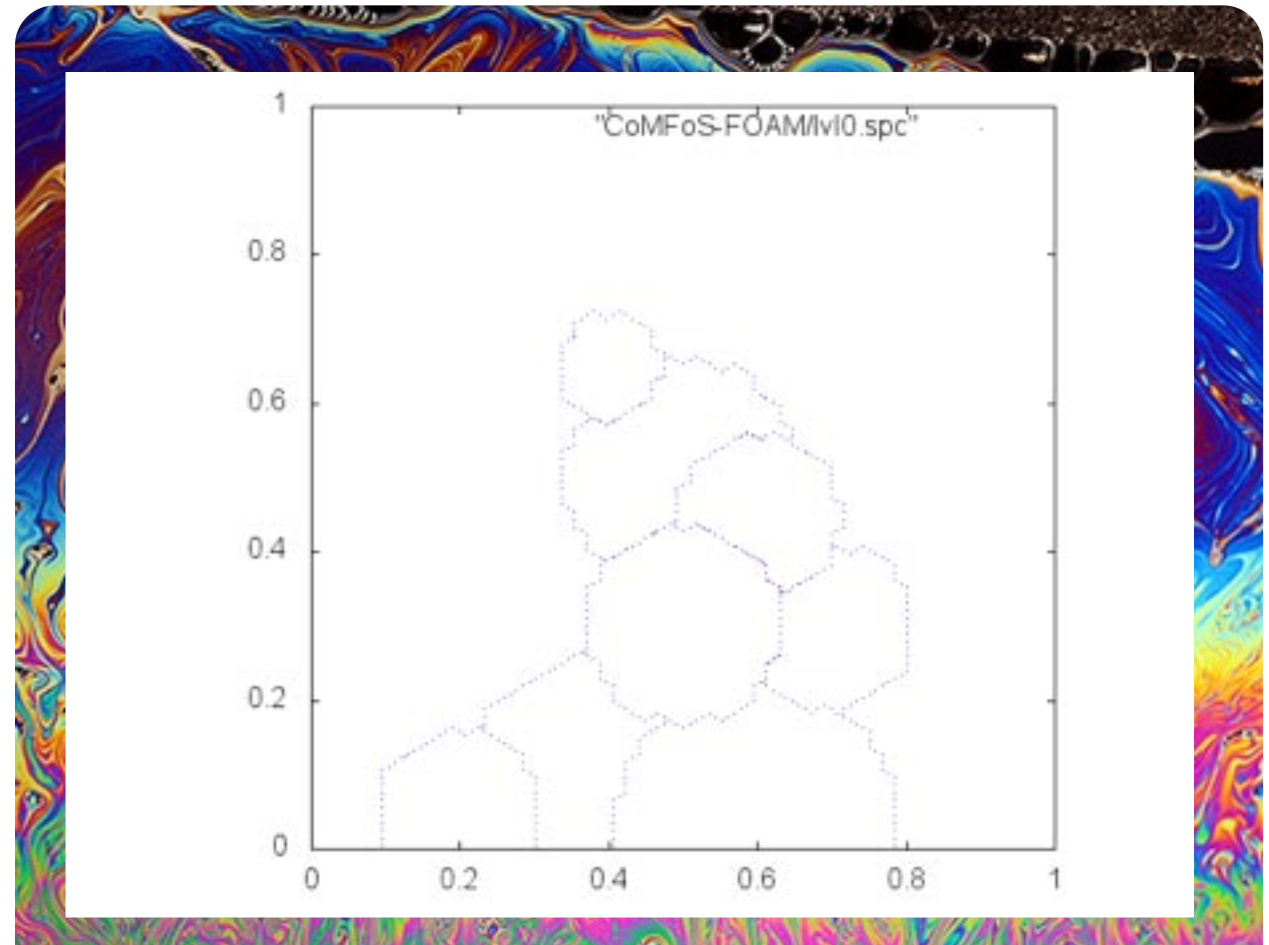


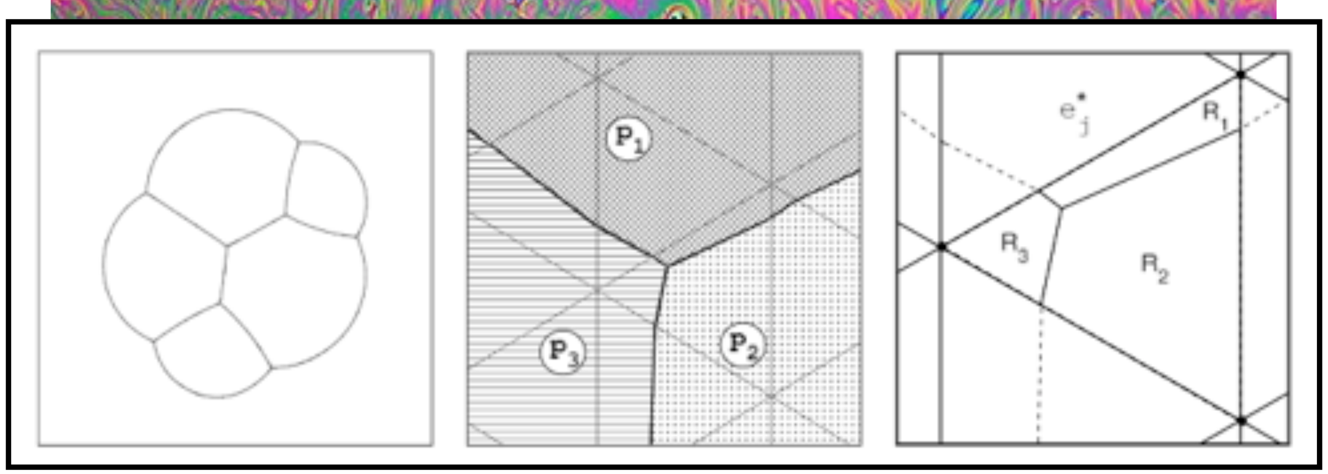


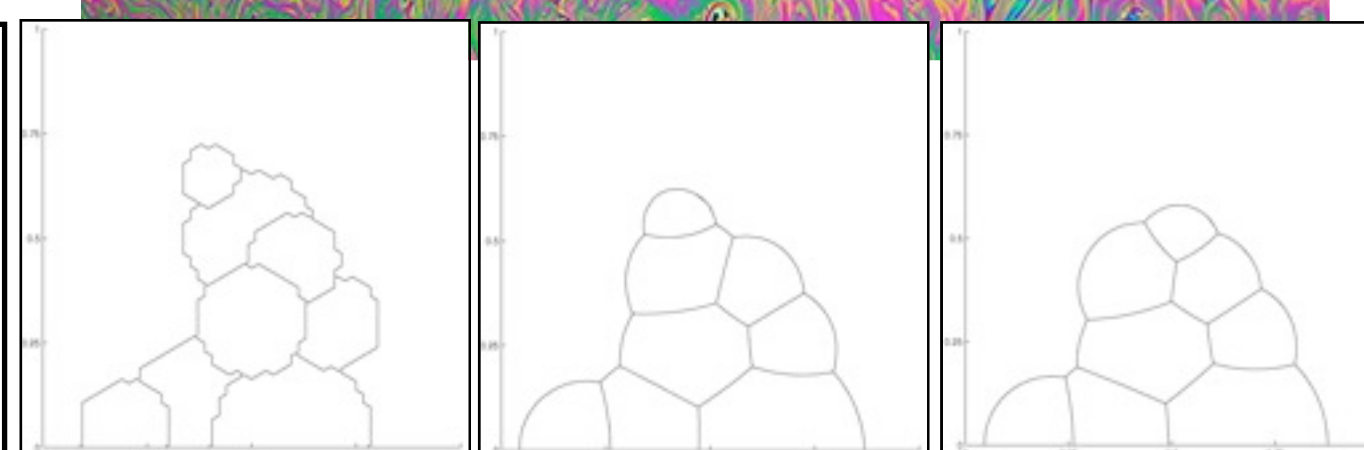


Multi-phase volume preserving









Hyperbolic Mean Curvature Flow

$$\begin{cases} \alpha''(t,s) = -\kappa(s) & (t,s) \in (0,T) \times [0,1) \\ \alpha'(t=0,s) = v_0(s) & s \in [0,1) \\ \alpha(t=0,s) = \gamma(s) & s \in [0,1) \end{cases}$$

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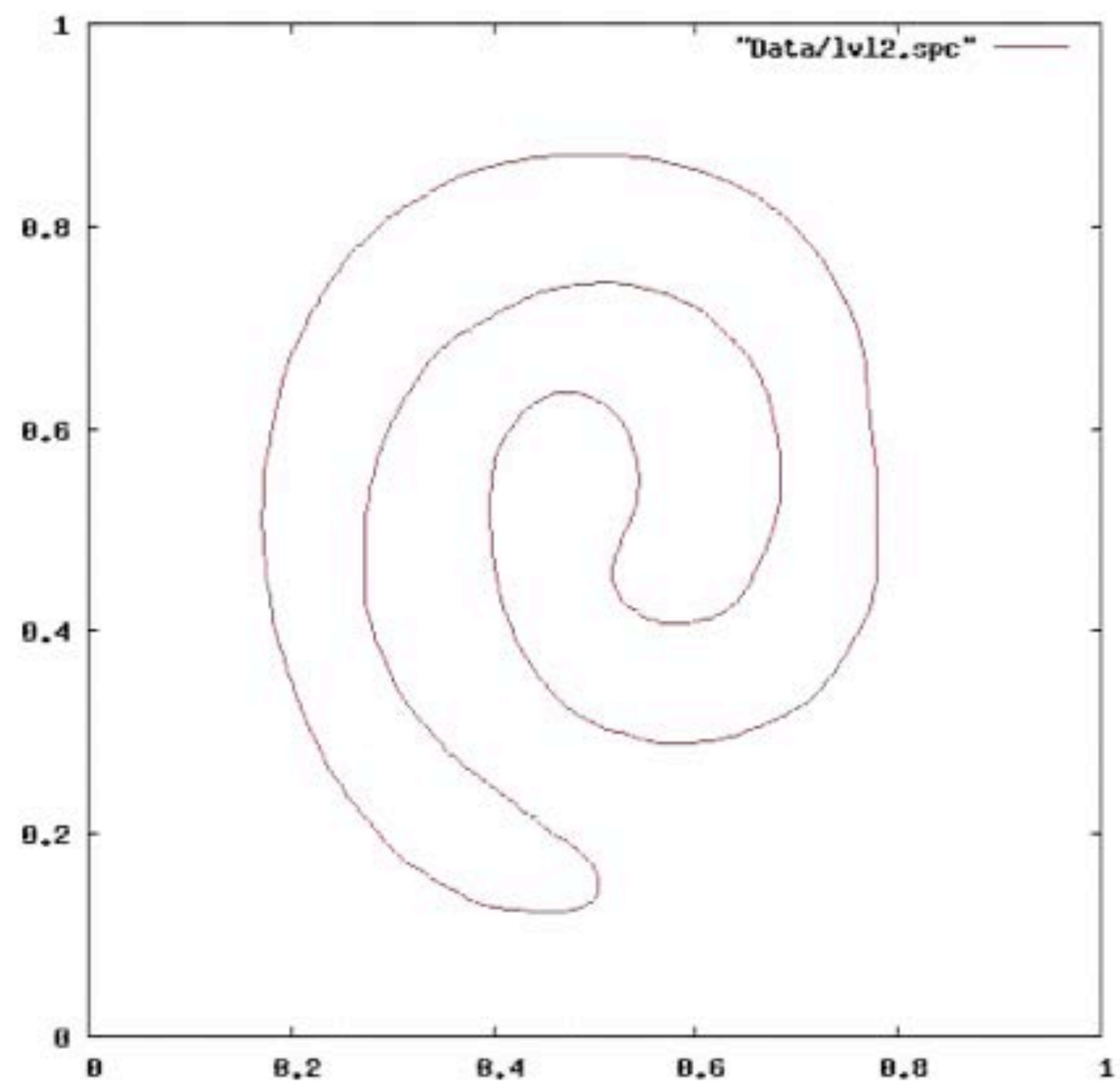
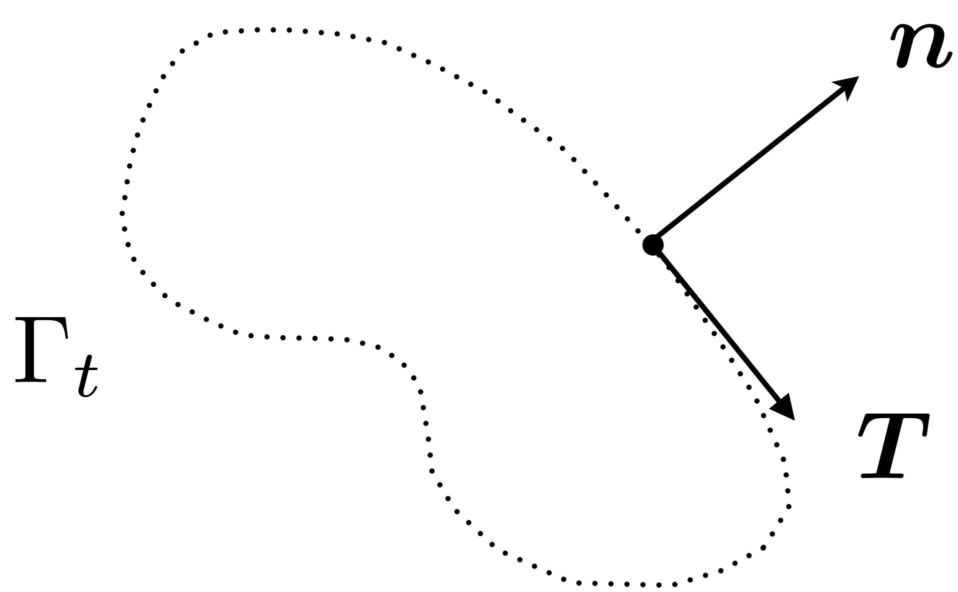


The geometrical setting

mean curvature flow

$v = -\kappa$

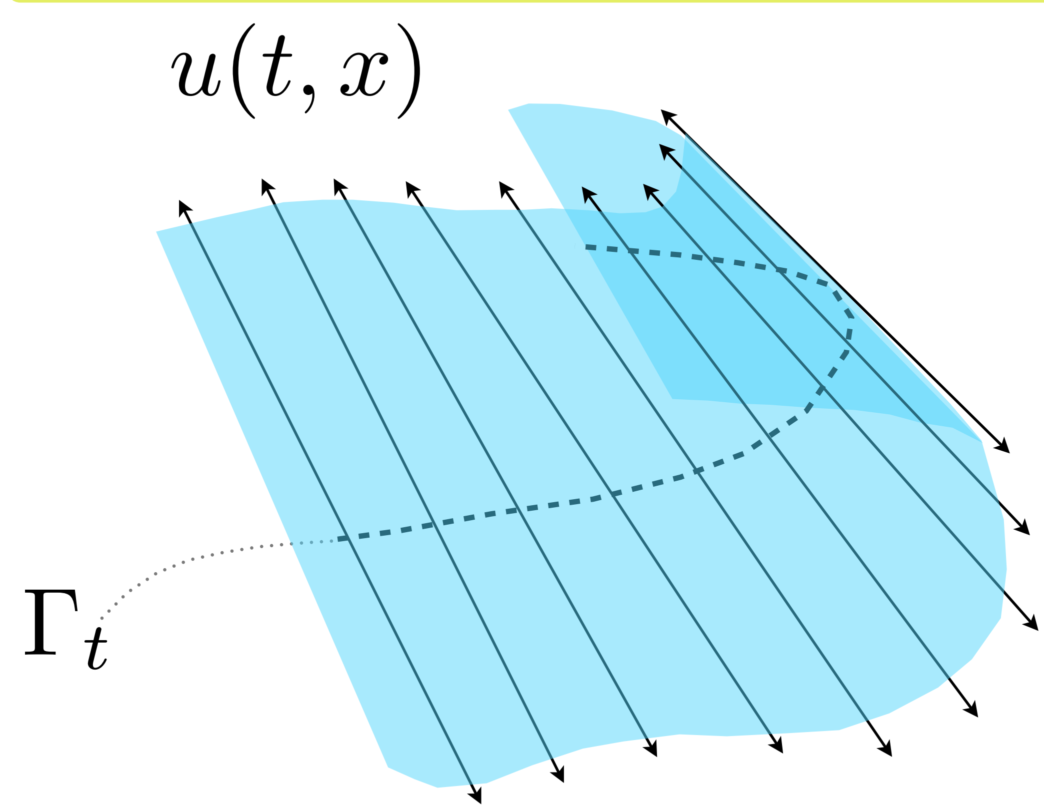


Γ_t : a closed curve at time t
$\boldsymbol{n}$: outer normal vector to Γ_t
$\boldsymbol{T}$: tangent vector to Γ_t
v : normal velocity
κ : mean curvature

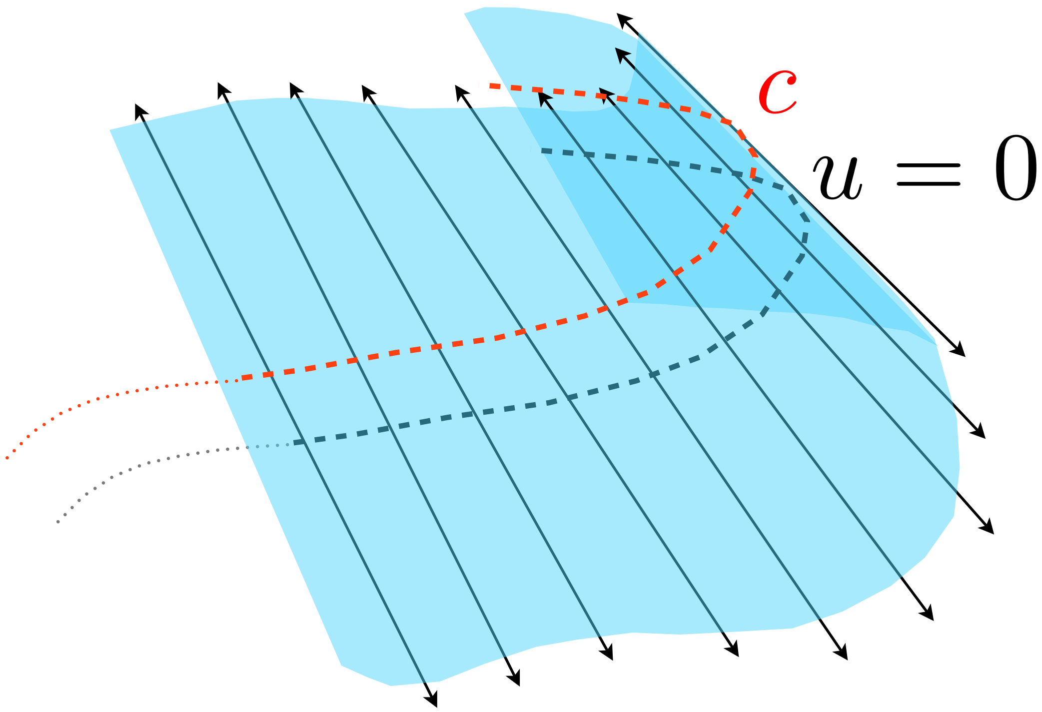
Encode the interface as a contour of a level set function $u(t,x)$

$$\Gamma_t = \{x : u(t,x) = 0\}$$

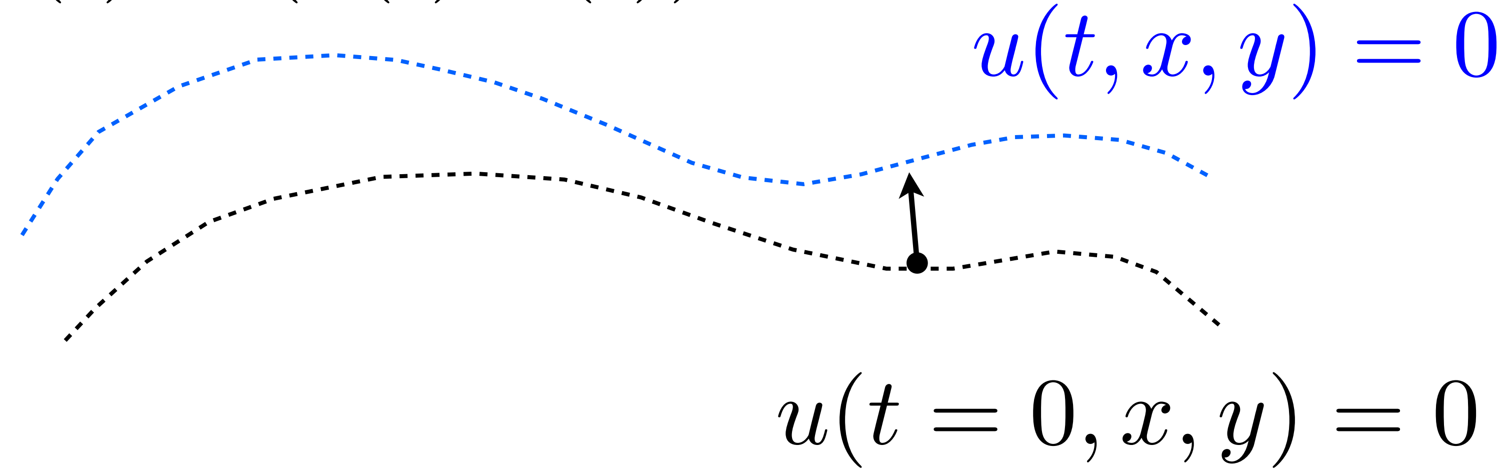
ex: signed distance



The idea (= follow the level set)



$$\boldsymbol{x}(t) = (x(t), y(t))$$



Contours of the level set function:

$$u(t, \boldsymbol{x}(t)) = c \qquad (c \in \mathbf{R})$$

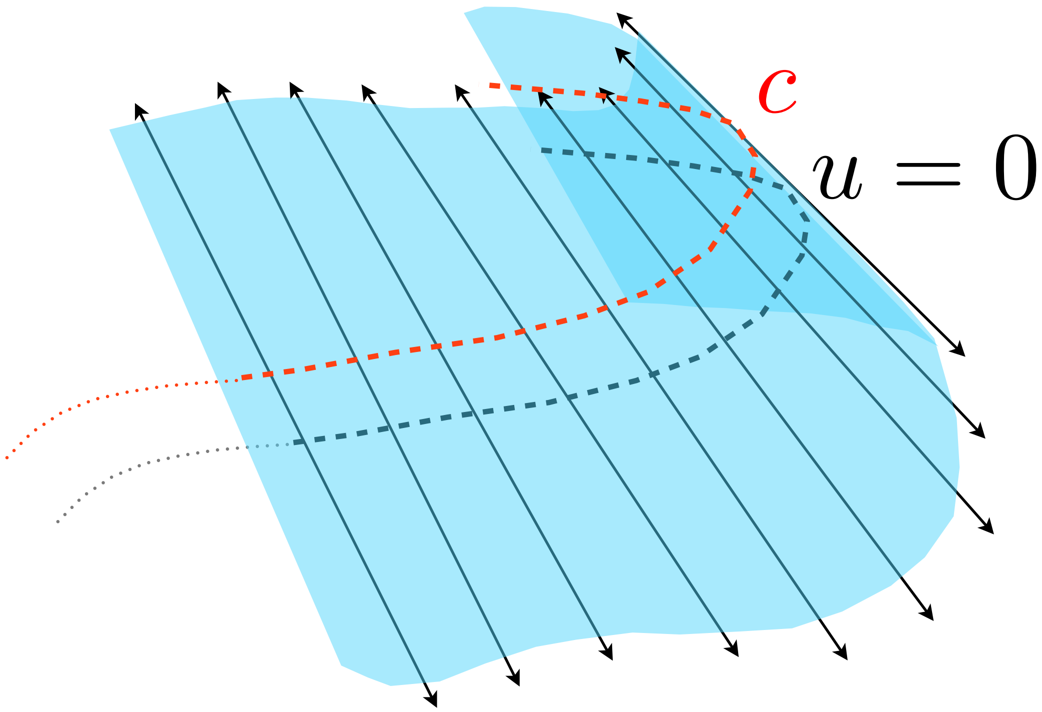
Encode the interface in the zero level set:

$$u(t, \boldsymbol{x}(t)) = 0$$

Follow the path of a particle along the level set

$$\frac{d}{dt} u(t, \boldsymbol{x}(t)) = 0 \implies u_t(t, x(t)) + \nabla u(t, x) \cdot \dot{\boldsymbol{x}}(t) = 0$$

The idea (= follow the level set)



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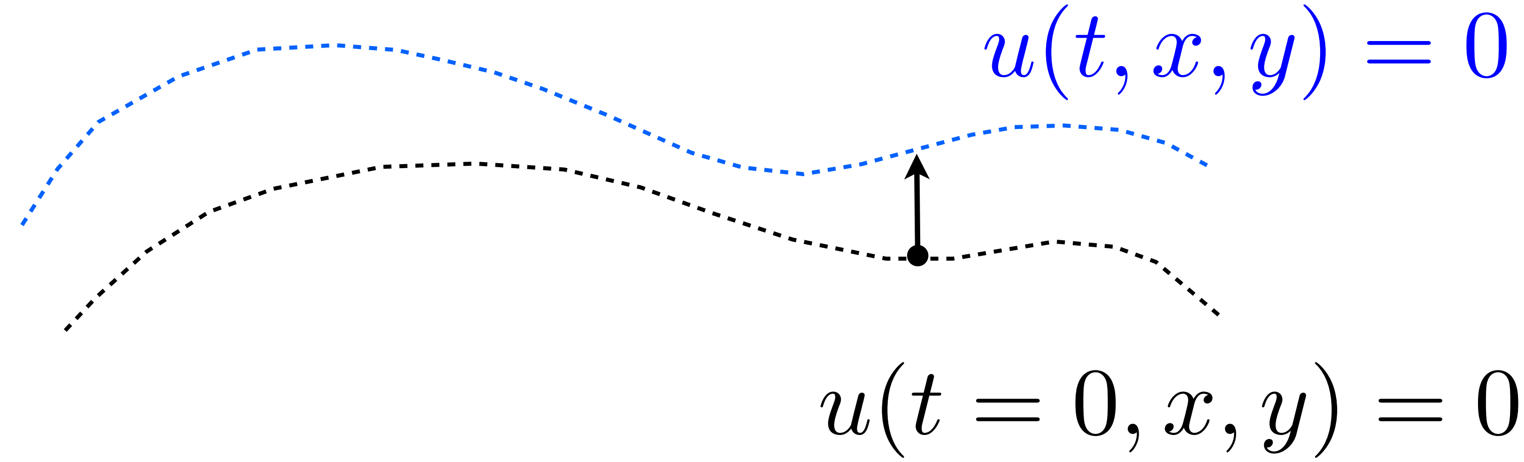


The velocity has a normal and tangental component:

$$\dot{\boldsymbol{x}}(t) = (V_{\boldsymbol{n}}\boldsymbol{n} + V_{\boldsymbol{T}}\boldsymbol{T}) \qquad u_t + \nabla u \cdot (V_{\boldsymbol{n}}\boldsymbol{n} + V_{\boldsymbol{T}}\boldsymbol{T}) = 0$$

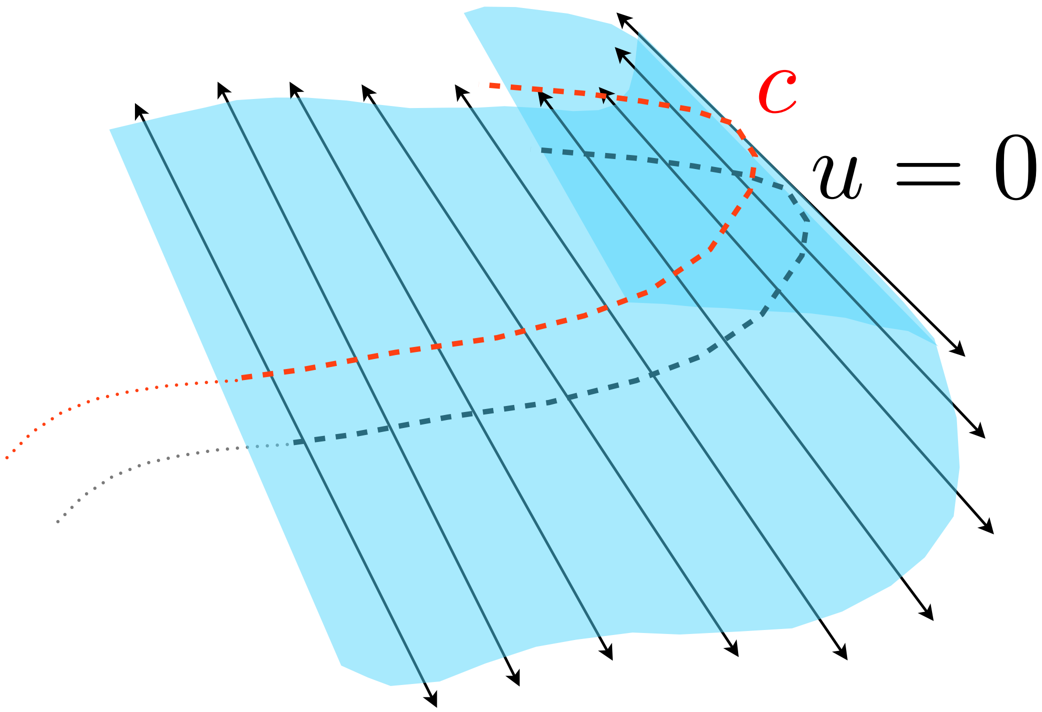
Rewrite using the normal vector:

$$u_t + |\nabla u|\, \boldsymbol{n} \cdot (V_{\boldsymbol{n}}\boldsymbol{n} + V_{\boldsymbol{T}}\boldsymbol{T}) = 0 \qquad \boldsymbol{n} = \frac{\nabla u}{|\nabla u|}$$

Using orthogonality relations:

$$u_t = -V_{\boldsymbol{n}}\, |\nabla u| \qquad \text{(the level set equation)}$$

The idea (= follow the level set)



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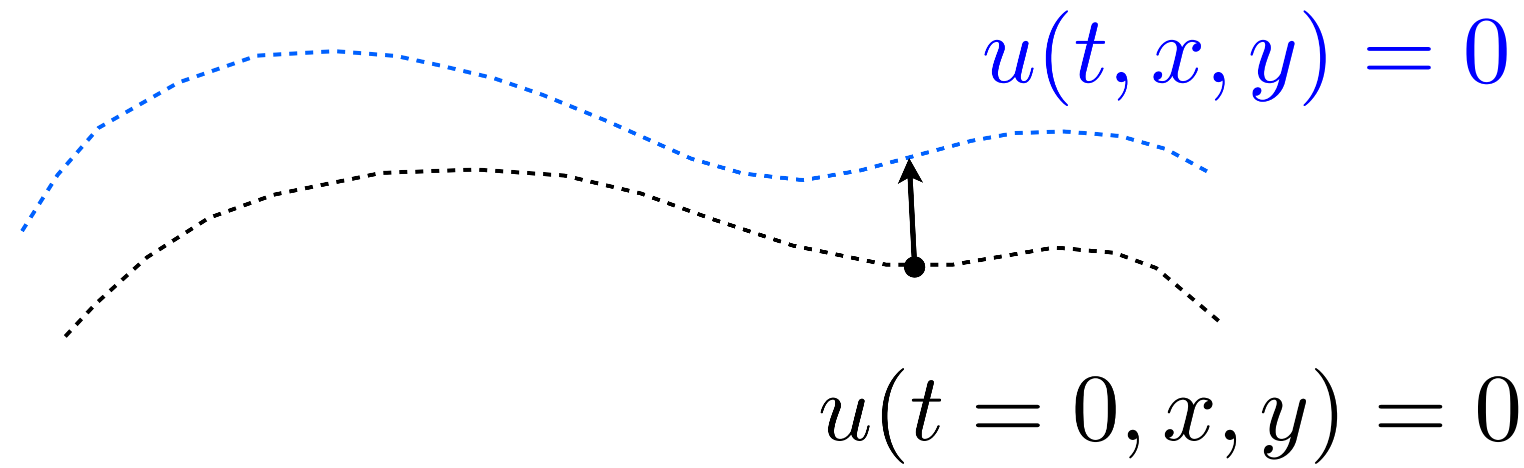


The interfacial velocity:

$$\frac{u_t}{|\nabla u|} = -V_{\boldsymbol{n}}$$

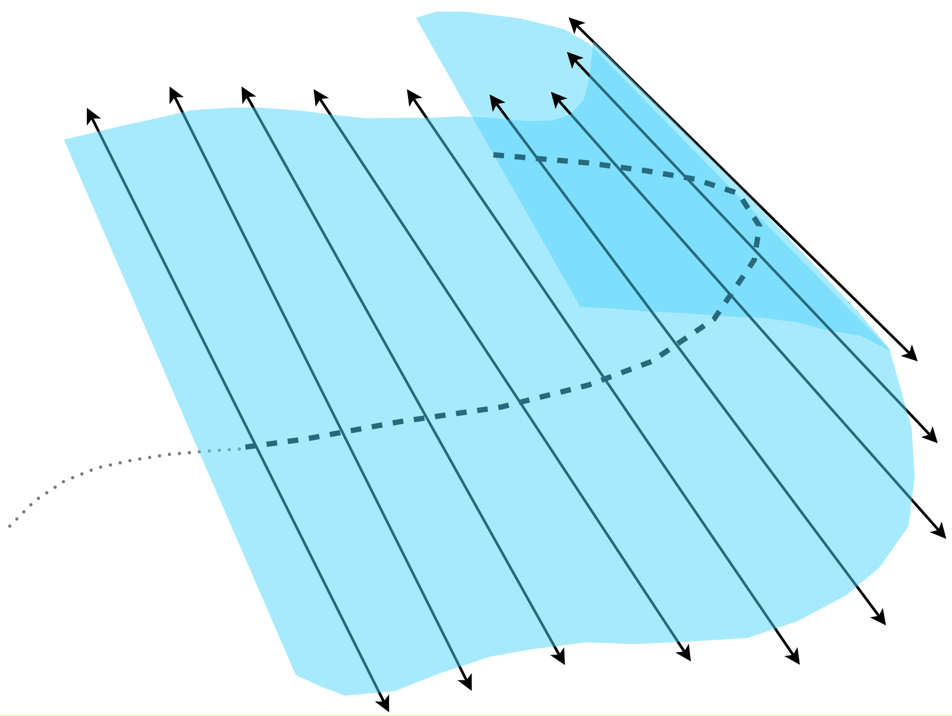
$$u_t = \Delta u \implies \text{BMO algorithm}$$

The interfacial acceleration:

$$\frac{\partial}{\partial t}\left[\frac{u_t}{|\nabla u|}\right] = -\dot{V}_{\boldsymbol{n}} \implies \frac{u_{tt}\,|\nabla u| - |\nabla u|_t\, u_t}{|\nabla u|^2} = -\dot{V}_{\boldsymbol{n}}$$

$$\Gamma_t = \{x \in \Omega : u(t,x) = 0\}$$

The signed distance function



$$d(t,x) = \begin{cases} \inf_{y\in\Gamma_t} ||x-y|| & \text{if } x \in \{u(t,x) > 0\} \\ -\inf_{y\in\Gamma_t} ||x-y|| & \text{otherwise} \end{cases}$$

satisfies the Eikonal equation:

$$|\nabla d| = 1 \qquad \text{a.e. } x$$

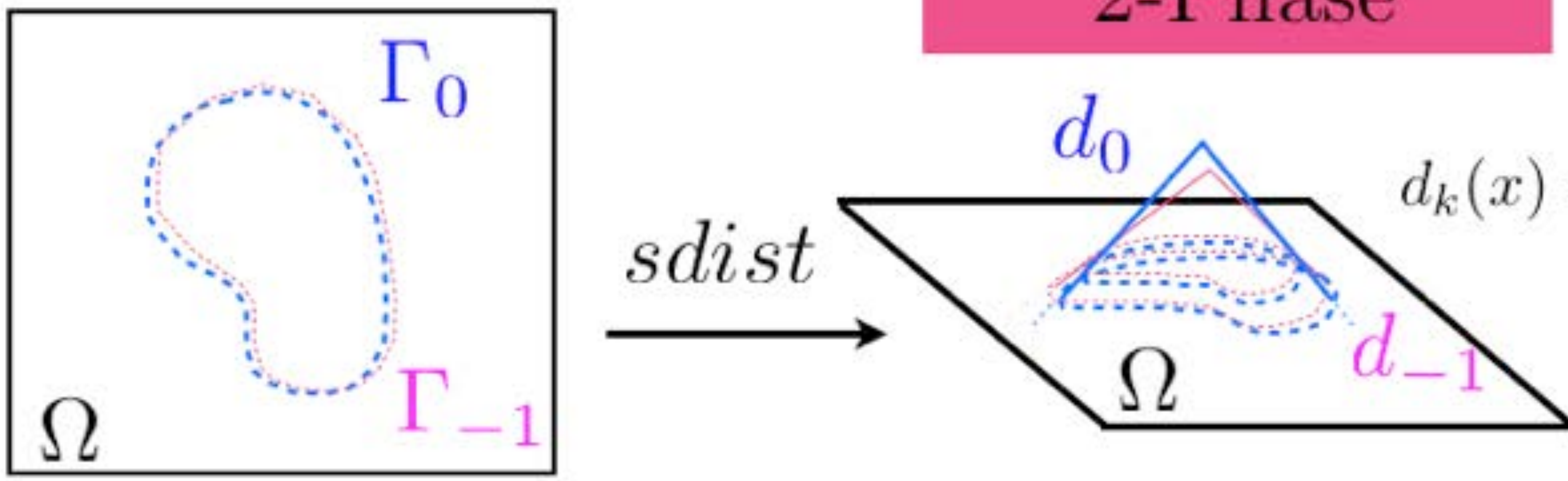
Choose the evolution: $u_{tt} = \Delta u$

$$-\dot{V}_{\boldsymbol{n}} = \frac{u_{tt}\,|\nabla u| - |\nabla u|_t\,u_t}{|\nabla u|^2} = \frac{\Delta u\,|\nabla u| - |\nabla u|_t\,u_t}{|\nabla u|^2} = \frac{(\nabla\cdot\nabla u)\,|\nabla u| - |\nabla u|_t\,u_t}{|\nabla u|^2}$$

$$= \frac{(\nabla\cdot(|\nabla u|\,\boldsymbol{n}))\,|\nabla u| - |\nabla u|_t\,u_t}{|\nabla u|^2} \implies \boxed{-\dot{V}_{\boldsymbol{n}} = \nabla\cdot\boldsymbol{n} = \kappa}$$

The hyperbolic BMO algorithm

Γ_0 Jordan curve



2-Phase

Signed distance function to Γ_0

$$d_k(x) = \begin{cases} \inf_{y\in\Gamma_k} ||x-y|| & \text{if } x \in E_k \\ -\inf_{y\in\Gamma_k} ||x-y|| & \text{otherwise} \end{cases}$$

Evolve the curve by $v' = -\kappa$ $\quad 0 < \Delta t \ll 1$

HBMO

for $k = 0, \ldots, MAX$ do

$$\begin{cases} u_{tt} = \Delta u & \text{in } (0, \Delta t) \times \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } (0, \Delta t) \times \partial\Omega \\ u(t=0, x) = 2d_k - d_{k-1} & \text{in } \Omega \\ u_t(t=0, x) = 0 & \text{in } \Omega \end{cases}$$

Solve the wave equation for a small time Δt

$$\Gamma_{k+1} = \partial\{u(t = \Delta t, x) > 0\}$$

Evolve the curve

$$d_{k+1}(x) = \begin{cases} \inf_{y\in\Gamma_{k+1}} ||x-y|| & \text{if } x \in \{u(t=\Delta t, x) > 0\} \\ -\inf_{y\in\Gamma_{k+1}} ||x-y|| & \text{otherwise} \end{cases}$$

Truncation

end

Theorem: Let $2d_k - d_{k-1}$ evolve by the wave equation $(\star)$, for a time $t > 0$. Then the zero level set of the solution evolves with normal acceleration: $v' = -\kappa + O(t)$ (and velocity $v(t) = v(0) - t\kappa + O(t^2)$).

$$\begin{cases} u_{tt} = \Delta u & \text{in } (0, \Delta t) \times \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } (0, \Delta t) \times \partial\Omega \\ u(t = 0, x) = 2d_k - d_{k-1} & \text{in } \Omega \\ u_t(t = 0, x) = 0 & \text{in } \Omega \end{cases} \qquad (\star)$$

Proof: (sketch, 2D) $x = (x_1, x_2)$

The solution to the wave equation with initial condition u_0 and initial velocity v_0 can be written:

$$u(x,t) = \frac{1}{2\pi t} \int_{B(x,t)} \frac{u_0(y) + t v_0(y) + \nabla u_0(y) \cdot (y - x)}{\sqrt{t^2 - |y - x|^2}} \, dy.$$

Consider an initial condition that is the sum of two signed distance functions:

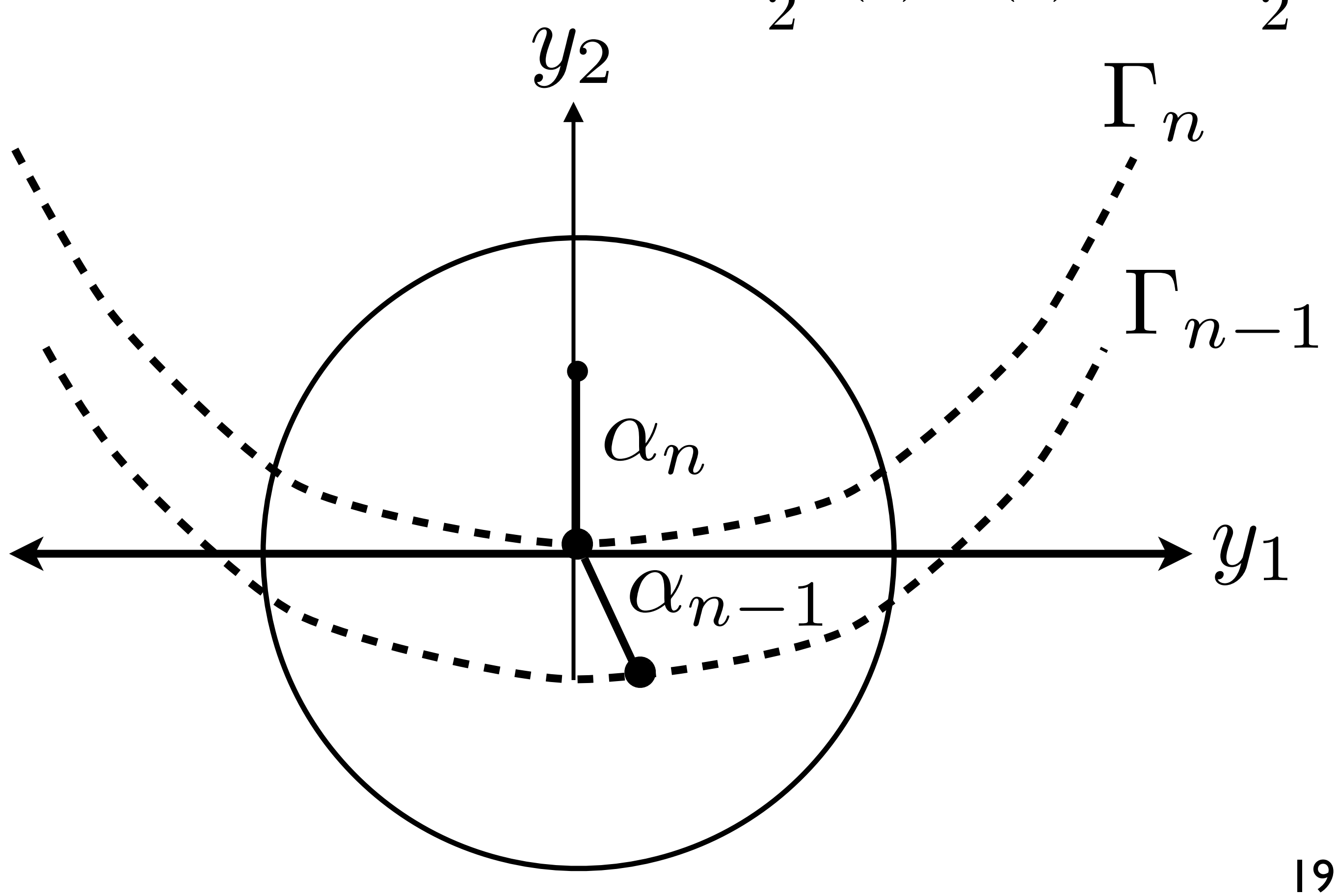
$$u_0 = a d_0 + b d_{-1}, v_0 = 0 \ (a, b, \in \mathbf{R})$$

A Hyperbolic BMO Algorithm: the justification

Theorem: (LeFloch et al., J. Math. Pures Appl., 2008, 90)
If the velocity field is normal to the hypersurface at time zero, then it is normal throughout the evolution.

Lemma: (Essedoglu et al., J. Comp. Phys., 2010, 229)
Let $f(x)$ be a smooth function whose graph $(x, f(x))$ describes the interface in a neighborhood of the origin. Then signed distance function $d(x, f(x))$ has the following Taylor expansion at $x = 0$:

(2-d):
$$d(x,y) = y + \frac{1}{2}\kappa(0)x^2 + \frac{1}{6}\kappa_x(0)x^3 - \frac{1}{2}\kappa^2(0)x^2y + \frac{1}{24}(\kappa_{xx}(0) - 3\kappa^3(0))x^4$$
$$-\frac{1}{2}\kappa(0)\kappa_x(0)x^3y + \frac{1}{2}\kappa^3(0)x^2y^2 + O(|\mathbf{x}|^5)$$



$$d_{n-1}(y) = y_2 + \alpha_{n-1} + \frac{1}{2}\kappa_{n-1}y_1^2 + O(|y|^3)$$

$$d_n(y) = y_2 + \frac{1}{2}\kappa_n y_1^2 + O(|y|^3)$$

Distance functions

Def. Given $A \subset \mathbf{R}^N$, the *distance function* from a point x to A is defined as:

$$d_A(x) = \inf_{y \in A} |y - x|.$$

Remark. Suppose $\partial\Omega$ is the boundary of a set Ω. Then

$$x \in \partial\Omega \implies d_{\partial\Omega}(x) = 0$$

Thm. The map $x \to d_A(x)$ is uniformly Lipschitz continuous in $\mathbf{R}^N$:

$$\forall x, y \in \mathbf{R}^N, \quad |d_A(y) - d_A(x)| \leq |y - x|.$$

proof. For all $z \in A$ and $x, y \in \mathbf{R}^N$

$$|z - y| \leq |z - x| + |y - x|$$
$$d_A(y) = \inf_{z \in A} |z - y| \leq \inf_{z \in A} |z - x| + |y - x| = d_A(x) + |y - x|.$$

Distance functions

Thm. The distance function satisfies the Eikonal equation:

$$|\nabla d_{\partial\Omega}(x)| = 1 \quad a.e.\ in\ \mathbf{R}^N.$$

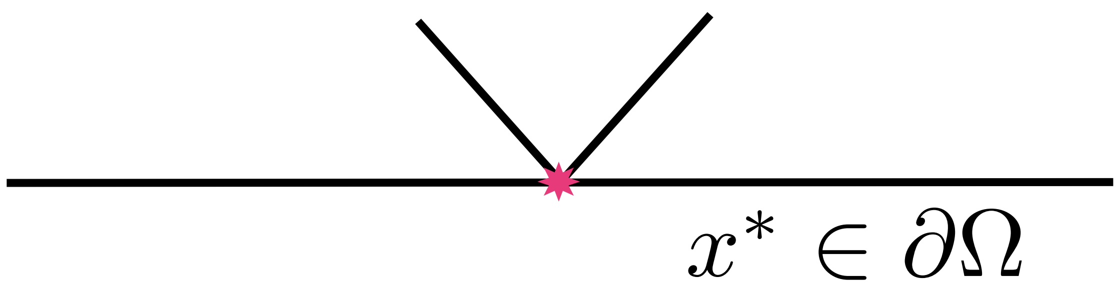
proof.

1. For a given location x, let x_c denote the point on $\partial\Omega$ that is closest to x.
2. Consider the line segment connecting x to x_c.
3. Note that x_c is the closest point on $\partial\Omega$ to every point along this segment.
4. $-\nabla d_{\partial\Omega}(x)$ gives the direction of steepest descent. $\blacksquare$

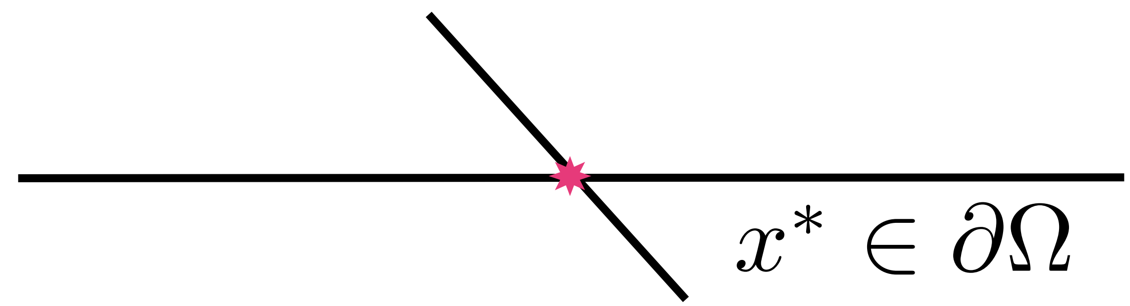
Distance functions

Since distance functions have "kinks" on their zero level set, *signed distance functions* are often used when we need to compute derivatives on $\partial\Omega$.

$$sign d_{\partial\Omega}(x) = \begin{cases} d(x, \partial\Omega) & x \in \Omega, \\ -d(x, \partial\Omega) & x \in \mathbf{R}^N \backslash \Omega. \end{cases}$$



$d_{\partial\Omega}(x)$



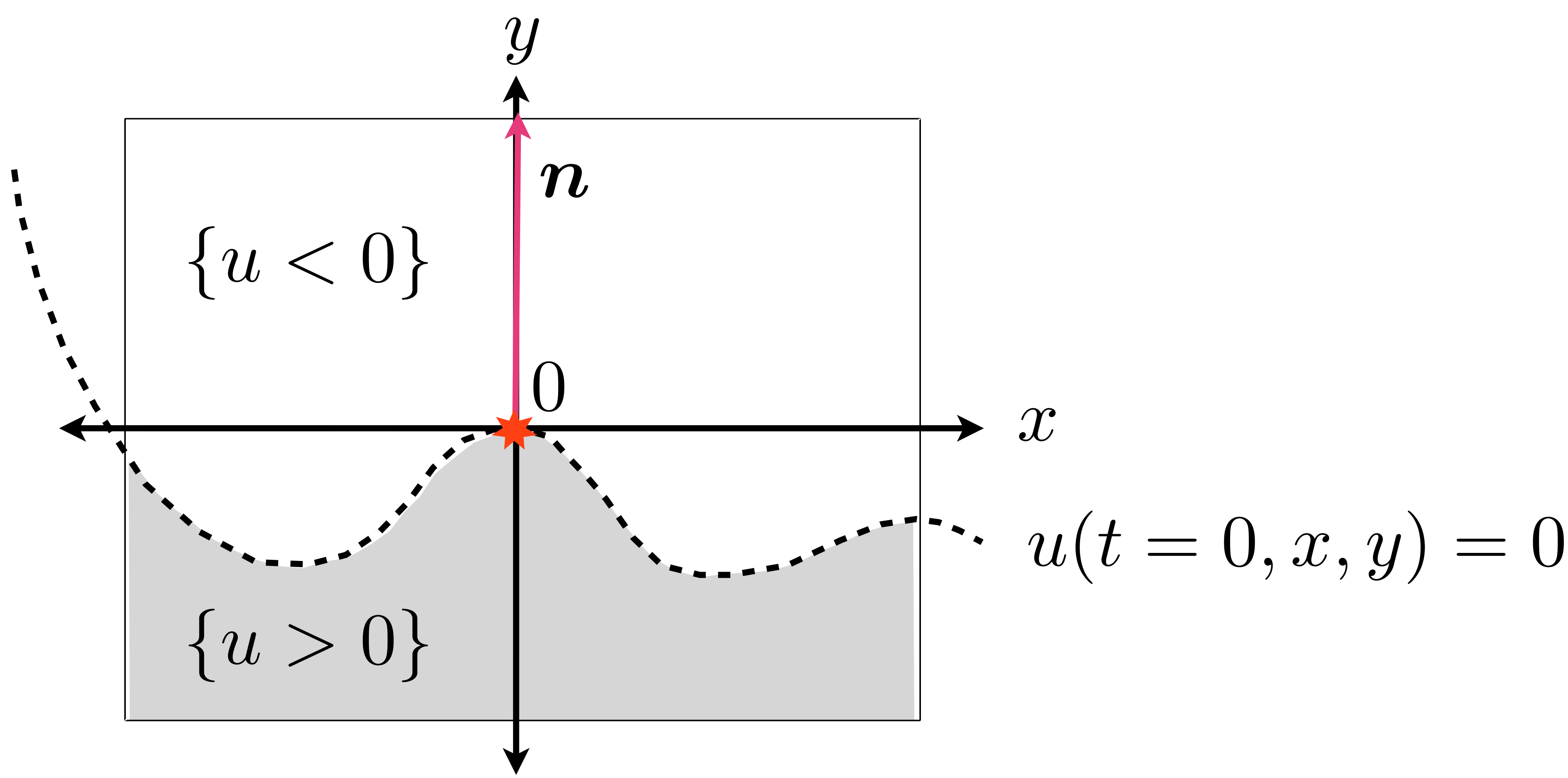
$sign d_{\partial\Omega}(x)$

Thm. The signed distance function $d(x, f(x))$ has the following Taylor expansion at $x = 0$:

$$d(x,y) = y + \frac{1}{2}\kappa(0)x^2 + \frac{1}{6}\kappa_x(0)x^3 - \frac{1}{2}\kappa^2(0)x^2y + \frac{1}{24}(\kappa_{xx}(0) - 3\kappa^3(0))x^4 - \frac{1}{2}\kappa(0)\kappa_x(0)x^3y + \frac{1}{2}\kappa^3(0)x^2y^2 + O(|\boldsymbol{x}|^5)$$

proof. (Essedoglu et al.)

This follows directly from the following four lemmas.



A Taylor expansion for the signed distance function

Lemma 1. For sufficiently small y, we have

$$d(0, y) = y.$$

$\Longrightarrow$

$$d_y(0, y) = 1 \quad \frac{\partial^k}{\partial y^k} d(0, y) = 0 \quad (\text{for } k = 2, 3, 4, ...) \quad \text{and} \quad d_x(0, y) = 0$$

proof.

From the previous discussion, we already have $d(0, y) = y$. The partial derivatives with respect to y yield the first two expressions, and the last expression then follows from the eikonal equation. ■

A Taylor expansion for the signed distance function

Lemma 2. The following hold:

$$\frac{\partial^k}{\partial y^k} d_x(0,y) = 0 \quad (k = 1, 2, \ldots)$$

for all sufficiently small (x,y).

proof.

Let $A(x,y) = d_x^2(x,y) + d_y^2(x,y)$. The eikonal equation implies $A(x,y) = 1$.

Differentiating with respect to x and y:

$$\frac{1}{2}\frac{\partial}{\partial x}A(x,y) = d_x(x,y)d_{xx}(x,y) + d_y(x,y)d_{xy}(x,y) = 0 \qquad (\star)$$

$$\frac{1}{2}\frac{\partial}{\partial y}A(x,y) = d_x(x,y)d_{xy}(x,y) + d_y(x,y)d_{yy}(x,y) = 0.$$

Evaluating at $x = 0$ and using the result of lemma 1, one has

$$d_{xy}(0,y) = 0 \text{ (for all small enough } y.)$$

Differentiation with respect to y yields the claim. $\blacksquare$

A Taylor expansion for the signed distance function

Lemma 3. The following hold:

$$d_{xx}(0,0) = \kappa(0),$$
$$d_{xxy}(0,0) = -\kappa^2(0)$$
$$d_{xxx}(0,0) = \kappa_x(0).$$

proof.

Along the interface, $d_{xx}(x, f(x)) + d_{yy}(x, f(x)) = \kappa(x)$. Evaluating at $x = 0$ and using lemma 1 yields the first claim. To obtain the second claim, we differentiate $(\star)$ with respect to x again:

$$\frac{1}{2}A_{xx}(x,y) = d_{xx}^2 + d_x d_{xxx} + d_{xy}^2 + d_y d_{xxy} = 0.$$

Evaluating at $(x, y) = (0, 0)$ gives $d_{xxy}(0,0) = -\kappa^2(0)$ (where we have used the previous results).

A Taylor expansion for the signed distance function

To obtain the last claim, we differentiate the expression for the Laplacian of the distance function with respect to x:

$$\frac{\partial}{\partial x}\left(d_{xx}(x,f(x))+d_{yy}(x,f(x))\right)=\kappa_x(x)$$

$$||$$

$$d_{xxx}(x,f(x))+d_{xxy}(x,f(x))f'(x)+d_{yyx}(x,f(x))+d_{yyy}(x,f(x))f'(x)$$

$$(f(0)=0, f'(0)=0, f''(0)=-\kappa(0))$$

$$\Longrightarrow$$

$$d_{xxx}(0,0)+d_{xxy}(0,0)f'(0)+d_{yyx}(0,0)+d_{yyy}(0,0)f'(0)=-\kappa_x(0)$$

■

A Taylor expansion for the signed distance function

Lemma 4. The following hold:

$$d_{xxxy}(0,0) = -3\kappa(0)\kappa_x(0),$$
$$d_{xxyy}(0,0) = 2\kappa^3(0),$$
$$d_{xxxx}(0,0) = \kappa_x x(0) - 3\kappa(0)^3.$$

proof.

Differentiating with respect to x:

$$\frac{1}{2}A_{xxx}(x,y) = \frac{\partial}{\partial x}\left(d_{xx}^2 + d_x d_{xxx} + d_{xy}^2 + d_y d_{xxy}\right) = 0$$
$$= 3d_{xx}d_{xxx} + 3d_{xy}d_{xxy} + d_x d_{xxxx} + d_y d_{xxyx}$$

Evaluated at $(x,y) = (0,0)$ yields

$$3\kappa(0)\kappa_x(0) + 0 + 0 + d_{xxyx}(0,0) = 0.$$

A Taylor expansion for the signed distance function

Differentiating with respect to y:

$$\frac{1}{2}A_{xxy}(x,y) = \frac{\partial}{\partial y}\left(d_{xx}^2 + d_x d_{xxx} + d_{xy}^2 + d_y d_{xxy}\right) = 0$$

$$= 2d_{xx}d_{xxy} + d_{xy}d_{xxx} + d_x d_{xxxy} + 2d_{xy}d_{xyy} + d_{yy}d_{xxy} + d_y d_{xxyy}$$

Evaluating at $(x,y) = (0,0)$:

$$0 = -2\kappa(0)\kappa^2(0) + 0 + 0 + 0 + 0 + d_{xxyy}(0,0).$$

$$\Longrightarrow \qquad 2\kappa^3(0) = d_{xxyy}(0,0).$$

A Taylor expansion for the signed distance function

Differentiating a previous result with respect to x again, we obtain:

$$\frac{\partial}{\partial x}\left(d_{xxx}(0,0) + d_{xxy}(0,0)f'(0) + d_{yyx}(0,0) + d_{yyy}(0,0)f'(0)\right) = -\kappa_{xx}(0)$$

$$\Longrightarrow$$

$$d_{xxxx}(x,f(x)) + d_{xxxy}(x,f(x))f'(x) + \left(d_{xxxy}(x,f(x)) + d_{xxyy}(x,f(x))f'(x)\right)f'(x) + f''(x)d_{xxy}(x,f(x))$$
$$+ d_{yyxx}(x,f(x)) + d_{yyyx}(x,f(x))f'(x) + \left(d_{yyyx}(x,f(x) + d_{yyyy}(x,f(x))f'(x)\right)f'(x) + f''(x)d_{yyy}(x,f(x))$$
$$= \kappa_{xx}(x)$$

Evaluated at $x = 0$, one obtains:

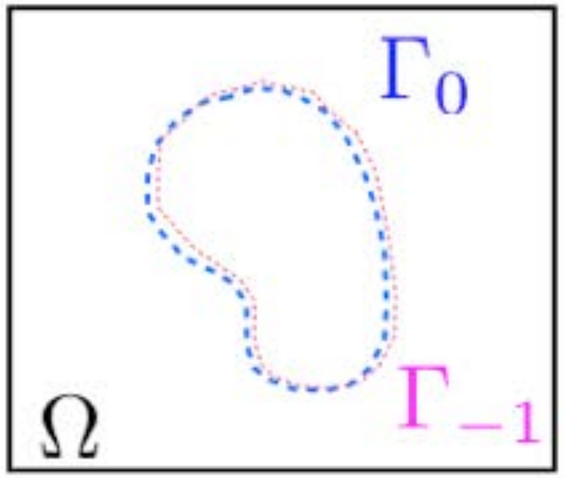
$$d_{xxxx}(0,0) + 0 + 0 + \kappa^3(0) + 2\kappa^3(0) + 0 + 0 + 0 = \kappa_{xx}(0).$$

$$\Longrightarrow$$

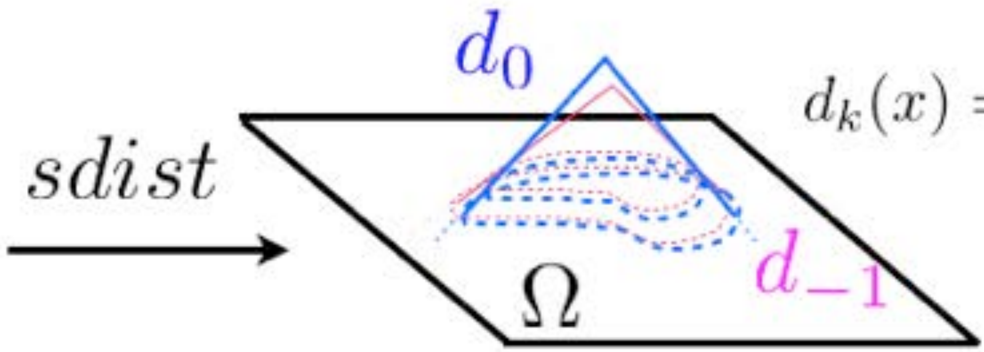
$$d_{xxxx}(0,0) = \kappa_{xx}(0) - 3\kappa^3(0) \qquad \blacksquare$$

The hyperbolic BMO algorithm

Γ_0 Jordan curve



2-Phase



Signed distance function to Γ_0

$$d_k(x) = \begin{cases} \inf_{y\in\Gamma_k} \|x-y\| & \text{if } x \in E_k \\ -\inf_{y\in\Gamma_k} \|x-y\| & \text{otherwise} \end{cases}$$

Evolve the curve by $\quad v' = -\kappa \qquad 0 < \Delta t \ll 1$

HBMO

for $k = 0, \ldots, MAX$ do

$$\begin{cases} u_{tt} = \Delta u & \text{in } (0, \Delta t) \times \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } (0, \Delta t) \times \partial\Omega \\ u(t=0, x) = 2d_k - d_{k-1} & \text{in } \Omega \\ u_t(t=0, x) = 0 & \text{in } \Omega \end{cases}$$

Solve the wave equation for a small time Δt

$$\Gamma_{k+1} = \partial\{u(t = \Delta t, x) > 0\}$$

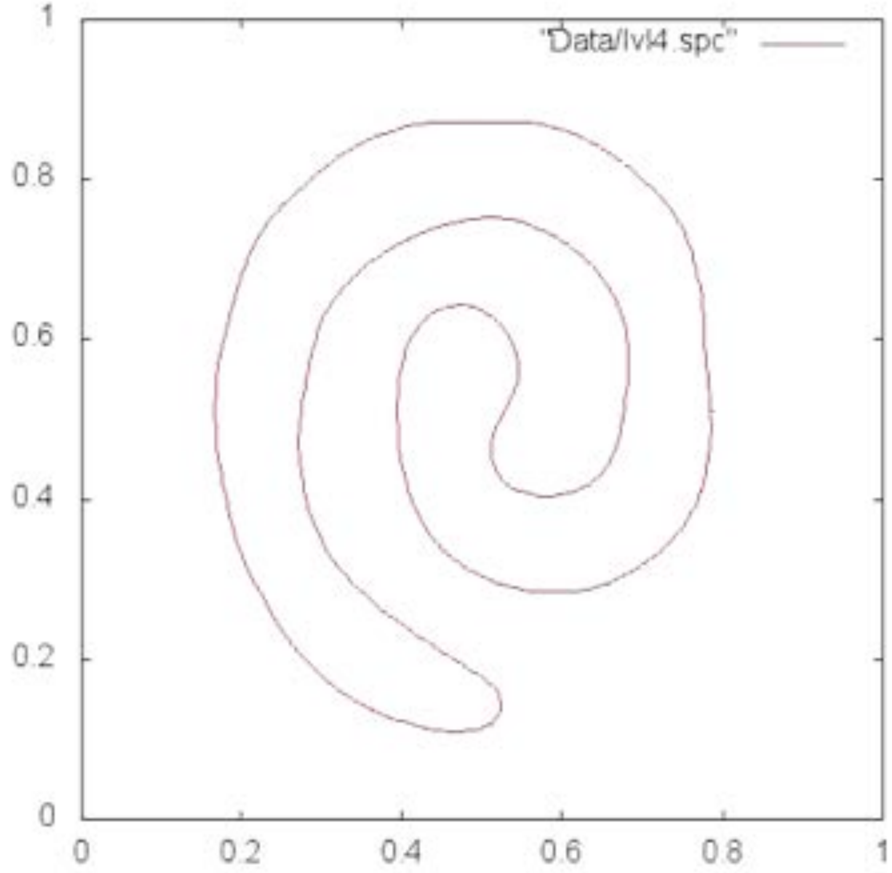
Evolve the curve

$$d_{k+1}(x) = \begin{cases} \inf_{y\in\Gamma_{k+1}} \|x-y\| & \text{if } x \in \{u(t=\Delta t, x) > 0\} \\ -\inf_{y\in\Gamma_{k+1}} \|x-y\| & \text{otherwise} \end{cases}$$

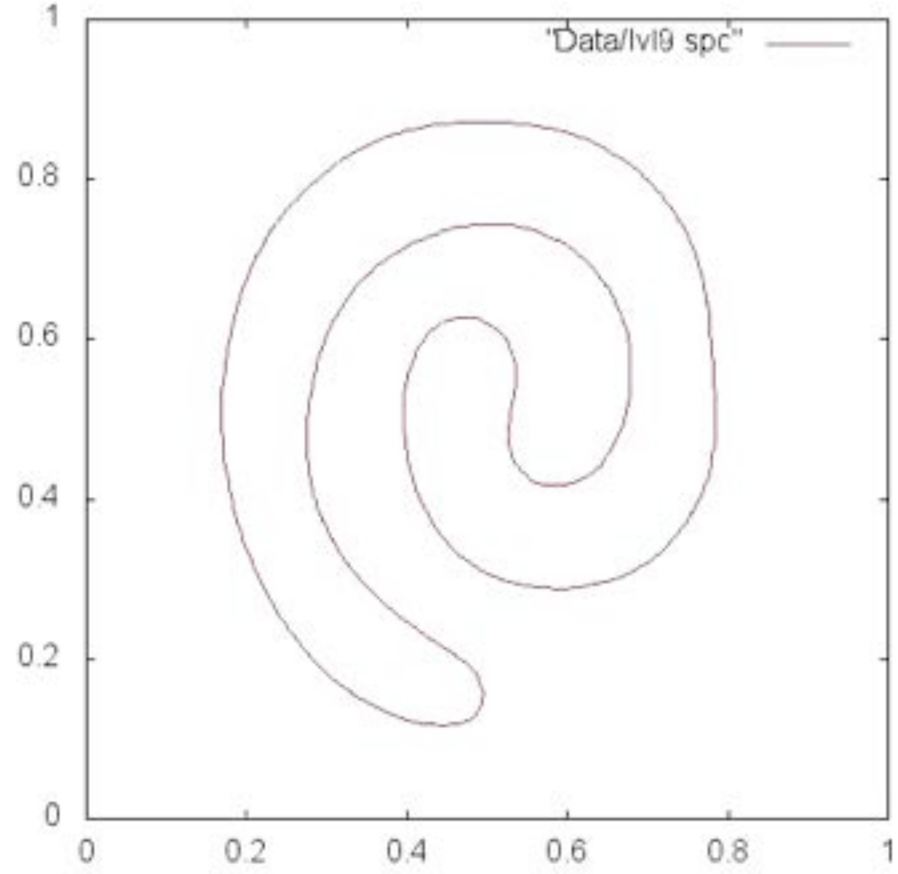
Truncation

end

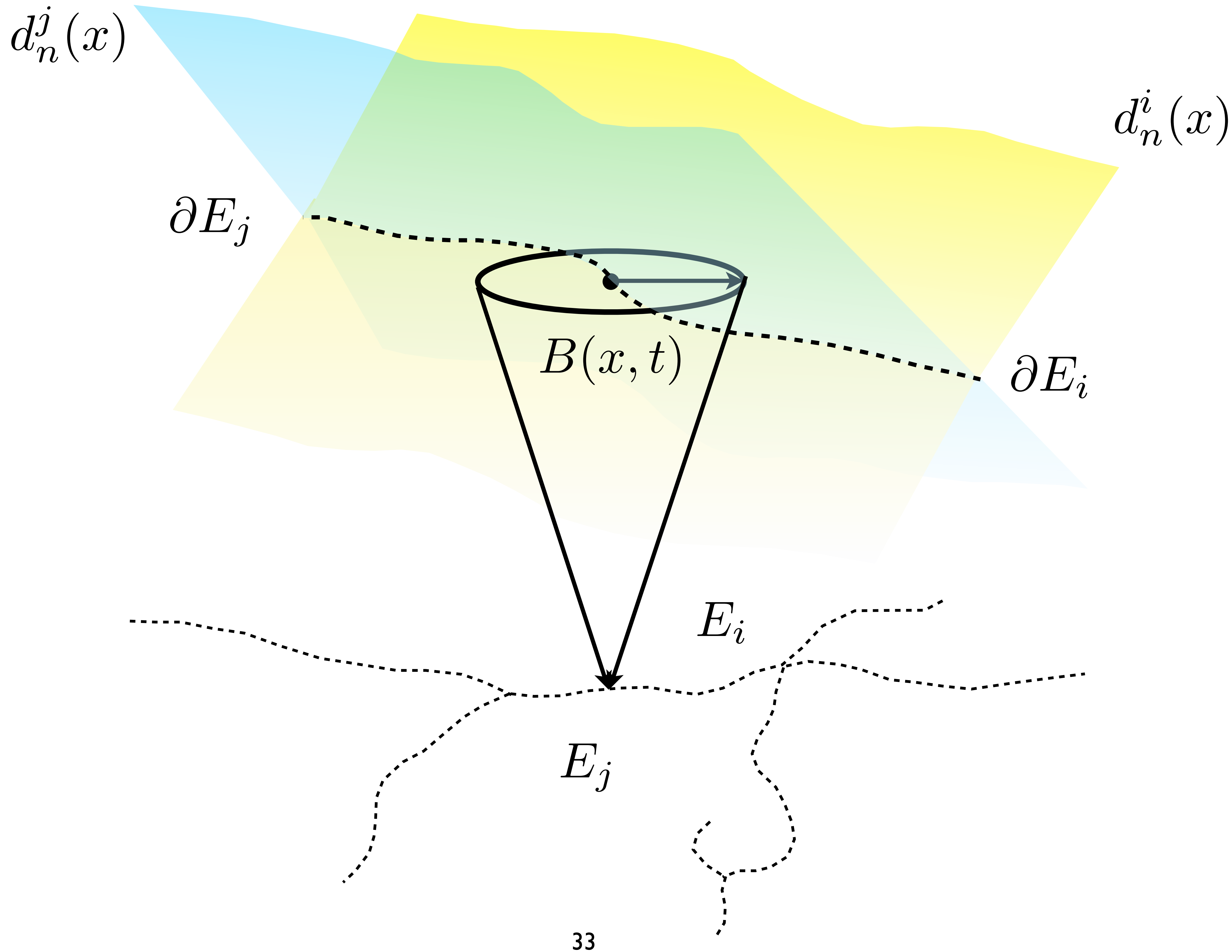
$$v = -\kappa$$



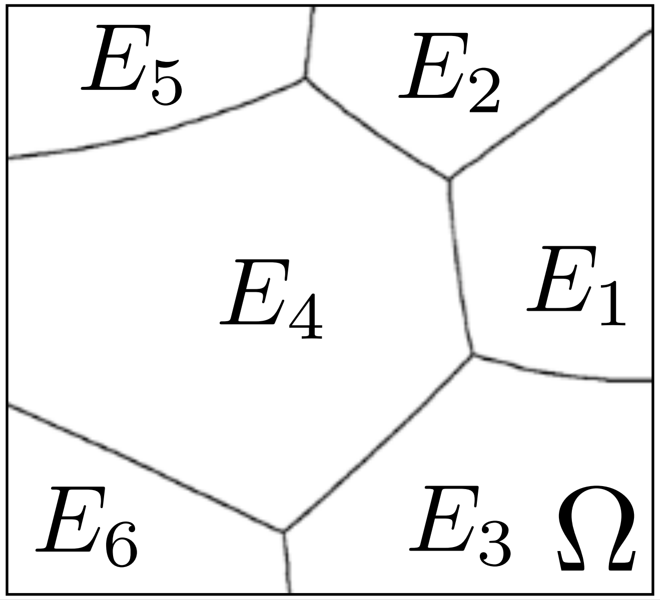
$$v' = -\kappa$$



Multiphase motions



Properties of wave propagation gives a multiphase algorithm



N : Number of phases

denote $d_i^k(x) = \begin{cases} dist(x, \partial E_i^k) & x \in E_i^k, \\ -dist(x, \partial E_i^k) & x \in \Omega \backslash E_i^k. \end{cases}$

$\Gamma_k^i = \partial E_i$: interface i at time $k\Delta t$

for $k = 1, ..., MAX$ do

$0 < \Delta t \ll 1$

for each $i = 1, ..., N$ evolve Γ_k^i by BMO

HBMO

$$\begin{cases} u_{tt}^i = \Delta u^i & \text{in } (0, \Delta t) \times \Omega \\ \dfrac{\partial u^i}{\partial \nu} = 0 & \text{on } (0, \Delta t) \times \partial\Omega \\ u^i(t = 0, x) = 2d_k^i - d_{k-1}^i & \text{in } \Omega \\ u_t^i(t = 0, x) = 0 & \text{in } \Omega \end{cases}$$

Solve the wave equation for a small time Δt

end

$E_i^k = \{x : u^i(\Delta t, x) > u^j(\Delta t, x), \forall j = 1, ..., N\}$

Truncation

$\Gamma_i^k = \partial E_i^k, i = 1, ..., N$

Evolve the curve

end

The reformulated multiphase hyperbolic BMO algorithm

HBMO

for $k = 1, ..., MAX$ do $\quad 0 < \Delta t \ll 1$

$i.$ Construct the initial vector field, z^{ϵ}_{k-1}

$ii.$ Solve

$$\begin{cases} \boldsymbol{u}_{tt} = \Delta \boldsymbol{u} & \text{in } (0, \Delta t) \times \Omega \\ \frac{\partial \boldsymbol{u}}{\partial \nu} = 0 & \text{on } (0, \Delta t) \times \partial\Omega \\ \boldsymbol{u}(t = 0, x) = 2\boldsymbol{z}^{\epsilon}_{k-1} - \boldsymbol{z}^{\epsilon}_{k-2} & \text{in } \Omega \\ \boldsymbol{u}_t(t = 0, x) = 0 & \text{in } \Omega \end{cases}$$

$iii.$ Evolve the interface:

$$E^k_i = \{x \in \Omega : \boldsymbol{u}(\Delta t, x) \cdot \boldsymbol{p}_i = \max_{j=1,...,N} \boldsymbol{u}(\Delta t, x) \cdot \boldsymbol{p}_j\}, i = 1, ..., N$$

$$\Gamma^k_i = \partial E^k_i, i = 1, ..., N$$

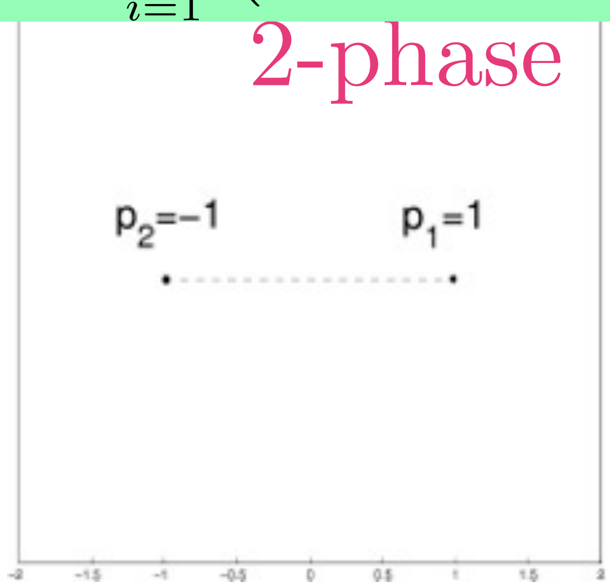
end

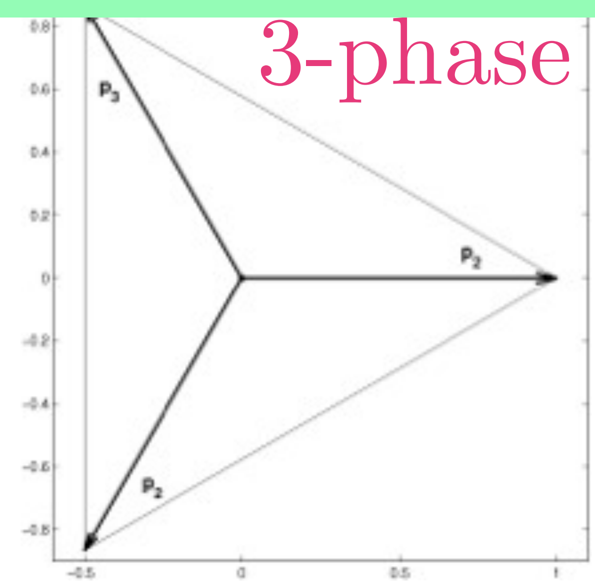
$$\boldsymbol{z}^{\epsilon}_k(x) = \sum_{i=1}^{N} \left(\boldsymbol{p}^i \chi_{\{d^k_i(x) \geq \frac{\epsilon}{2}\}} + \frac{1}{\epsilon}\left(\frac{\epsilon}{2} + d^k_i(x)\right) \boldsymbol{p}^i \chi_{\{\frac{-\epsilon}{2} < d^k_i(x) < \frac{\epsilon}{2}\}} \right)$$

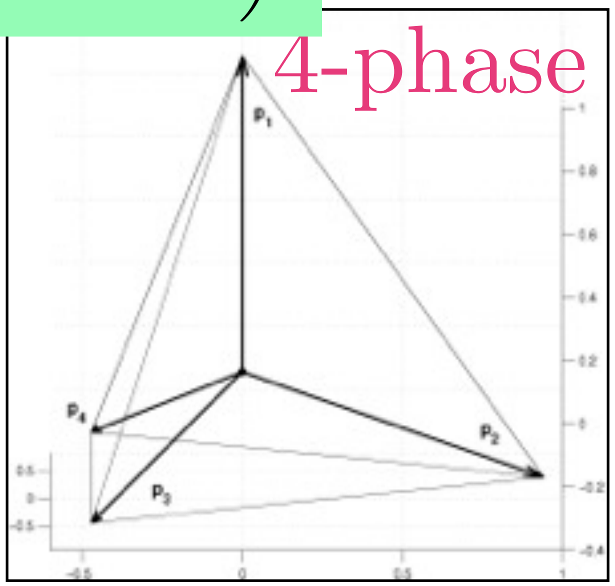
$\epsilon > 0$

$\boldsymbol{p}_k \in \mathbf{R}^{N-1}$:

coordinates are given by the vertices of a *regular simplex* in $\mathbf{R}^N$.

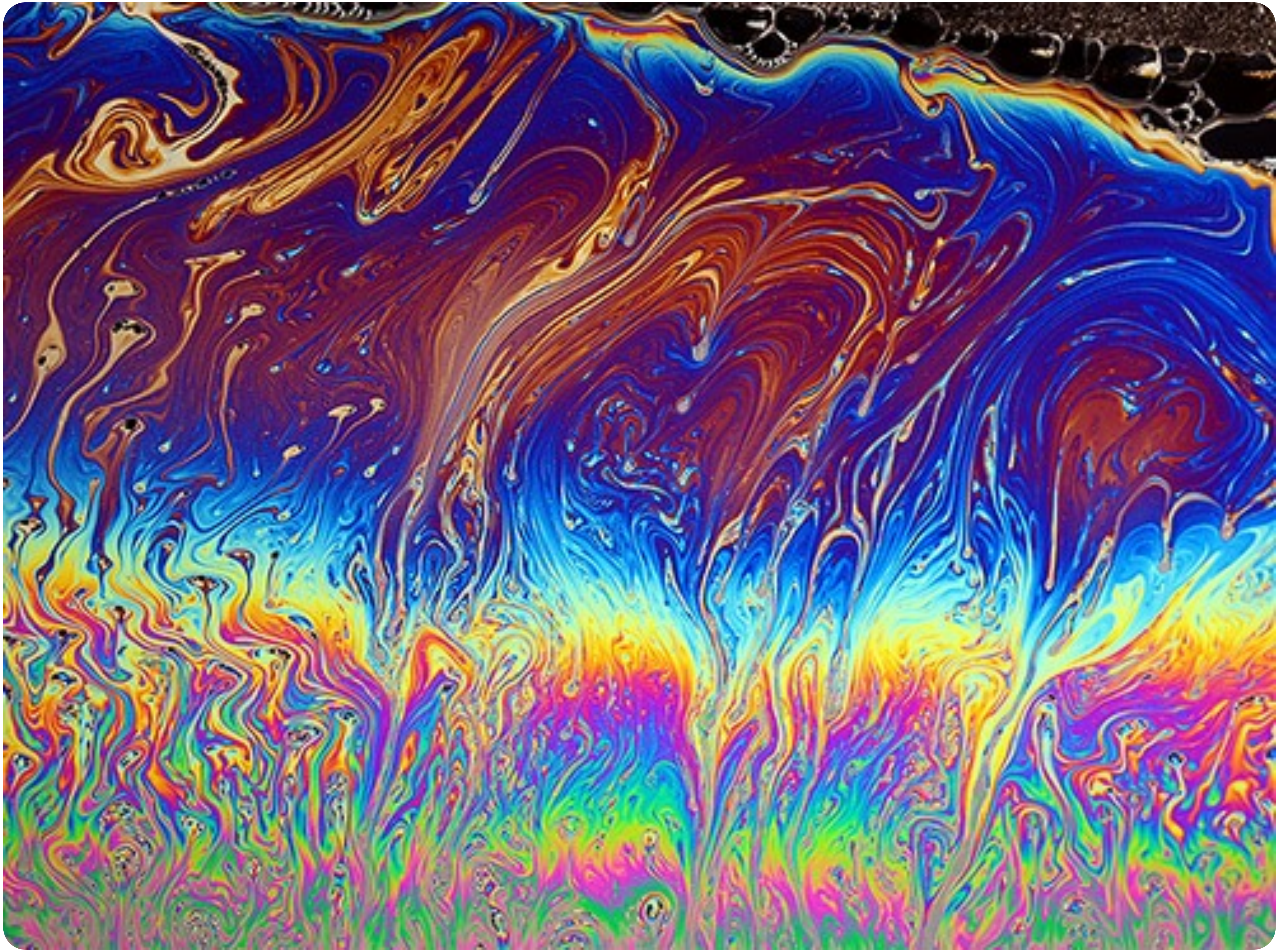






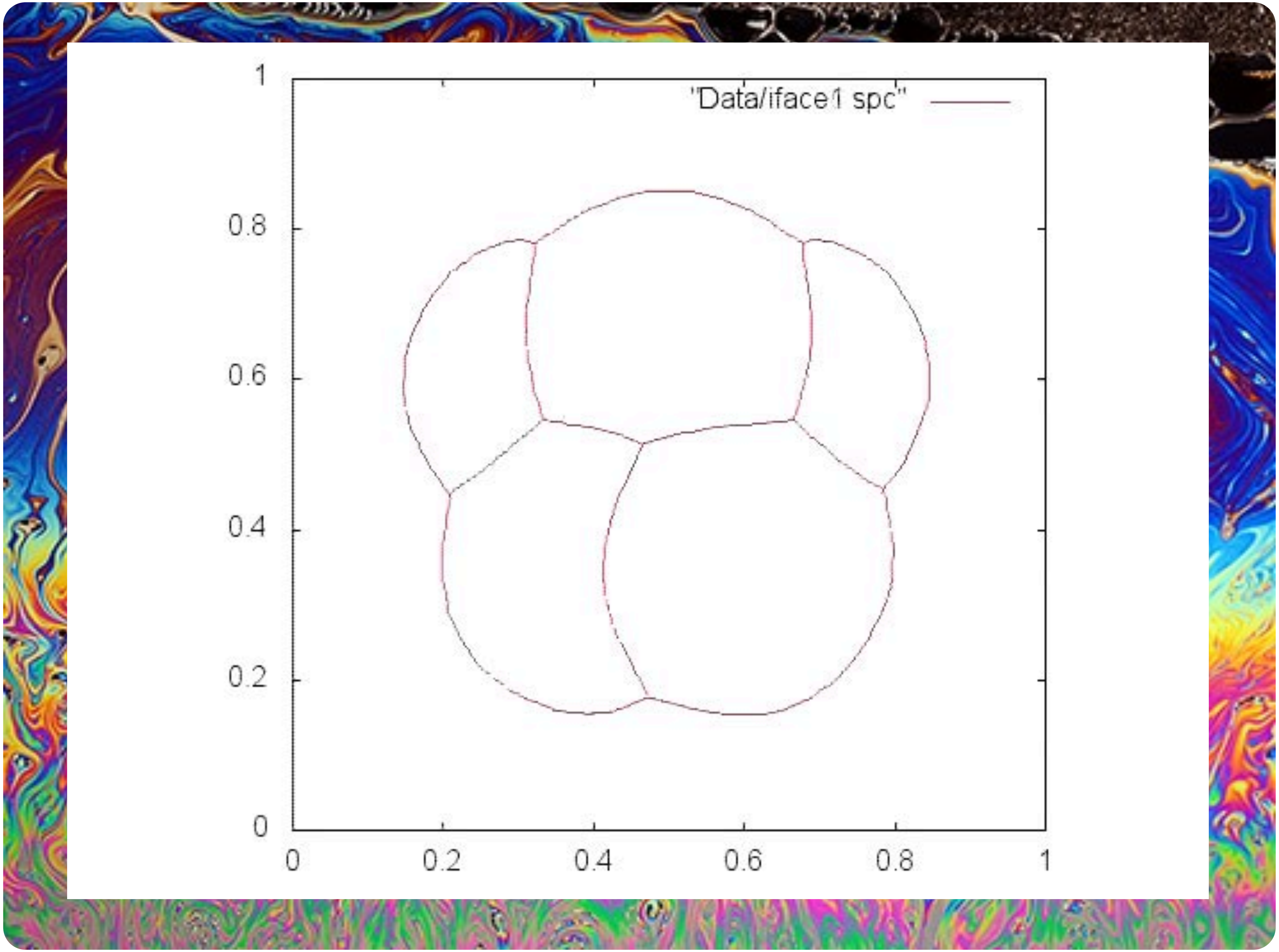
Numerical behavior of our approximation method

multiphase hyperbolic mean curvature flow



Numerical behavior of our approximation method

multiphase hyperbolic mean curvature flow



Vector-type minimizing movement: $\boldsymbol{u}, \boldsymbol{u}_{n-1}, \boldsymbol{u}_{n-2} \in H^1(\Omega; \mathbf{R}^{N-1})$

$$\mathcal{F}_n(\boldsymbol{u}) = \int_\Omega \left(\frac{|\boldsymbol{u} - 2\boldsymbol{u}_{n-1} + \boldsymbol{u}_{n-2}|^2}{2h^2} + \frac{|\nabla \boldsymbol{u}|^2}{2} \right) dx$$

$$+ \sum_{k=1}^{N-1} \frac{1}{\tilde{\epsilon}} |V_k - meas(E_k^n)|^2$$

Wave Type

$\tilde{\epsilon} > 0$

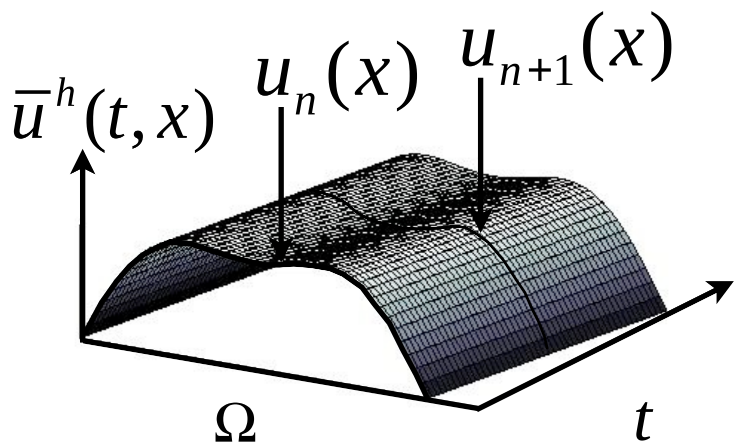
Penalties

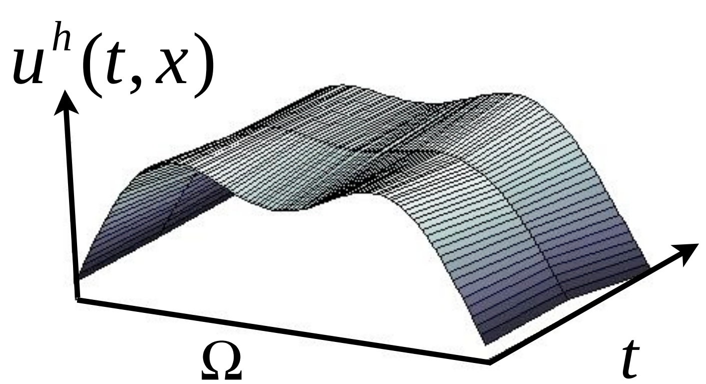
If there is no penalty, the minimizing movement converges:

$$u_t^h \overset{*}{\rightharpoonup} u_t, \qquad \nabla \bar{u}^h \overset{*}{\rightharpoonup} \nabla u, \qquad \nabla u^h \overset{*}{\rightharpoonup} \nabla u \qquad (\text{weakly star in } L^\infty(0,T; L^2(\Omega)))$$

$$\bar{u}^h \to u, \qquad u^h \to u \qquad (\text{strongly in } L^2(Q_T)).$$

$$\frac{u_t^h(t) - u_t^h(t-h)}{h} \overset{*}{\rightharpoonup} u_{tt}, \qquad (\text{weakly star in } L^\infty(0,T; L^2(\Omega)))$$

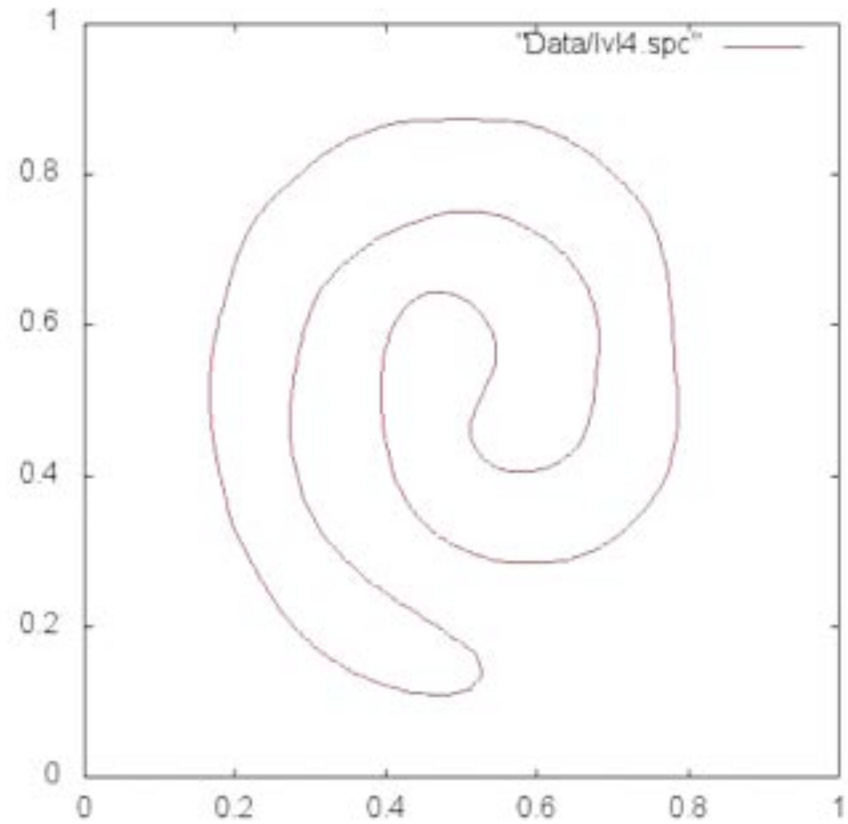




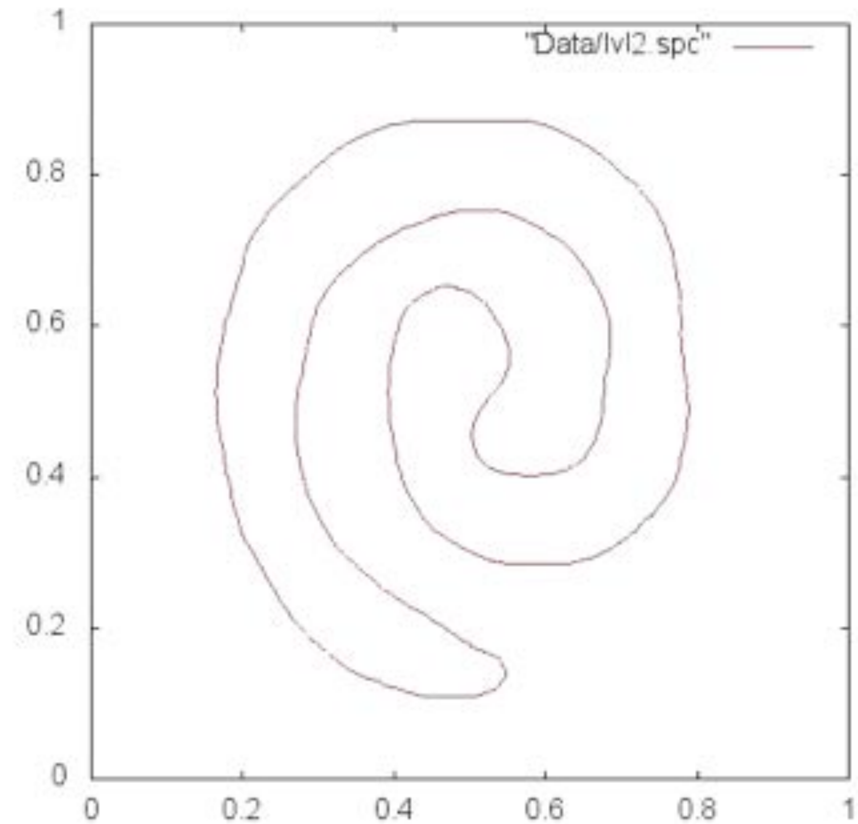
$$v = -\kappa + \bar{k}$$

$$v' = -\kappa + \bar{v}$$

$$v = -\kappa + \bar{k}$$

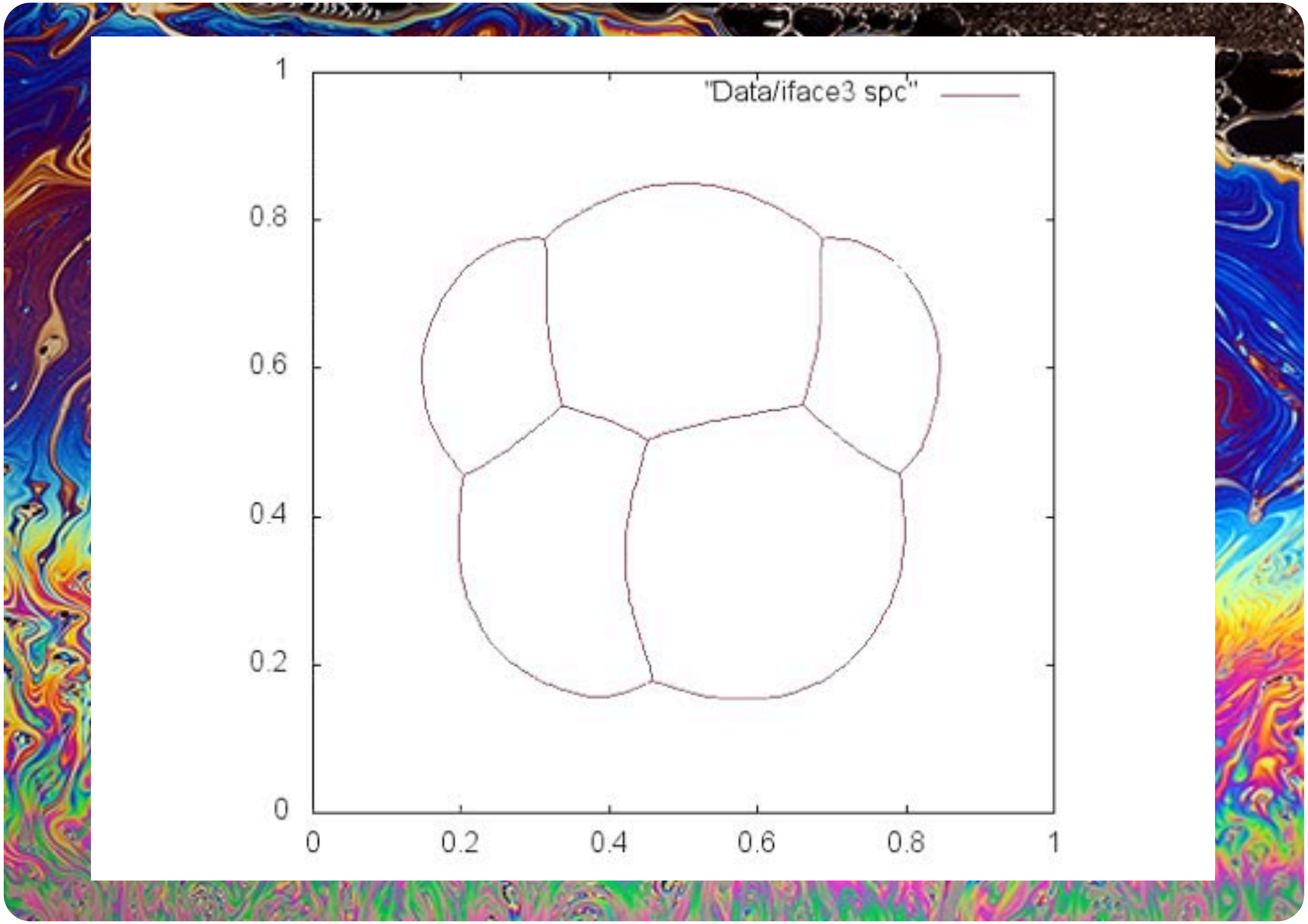


$$v' = -\kappa + \bar{v}$$



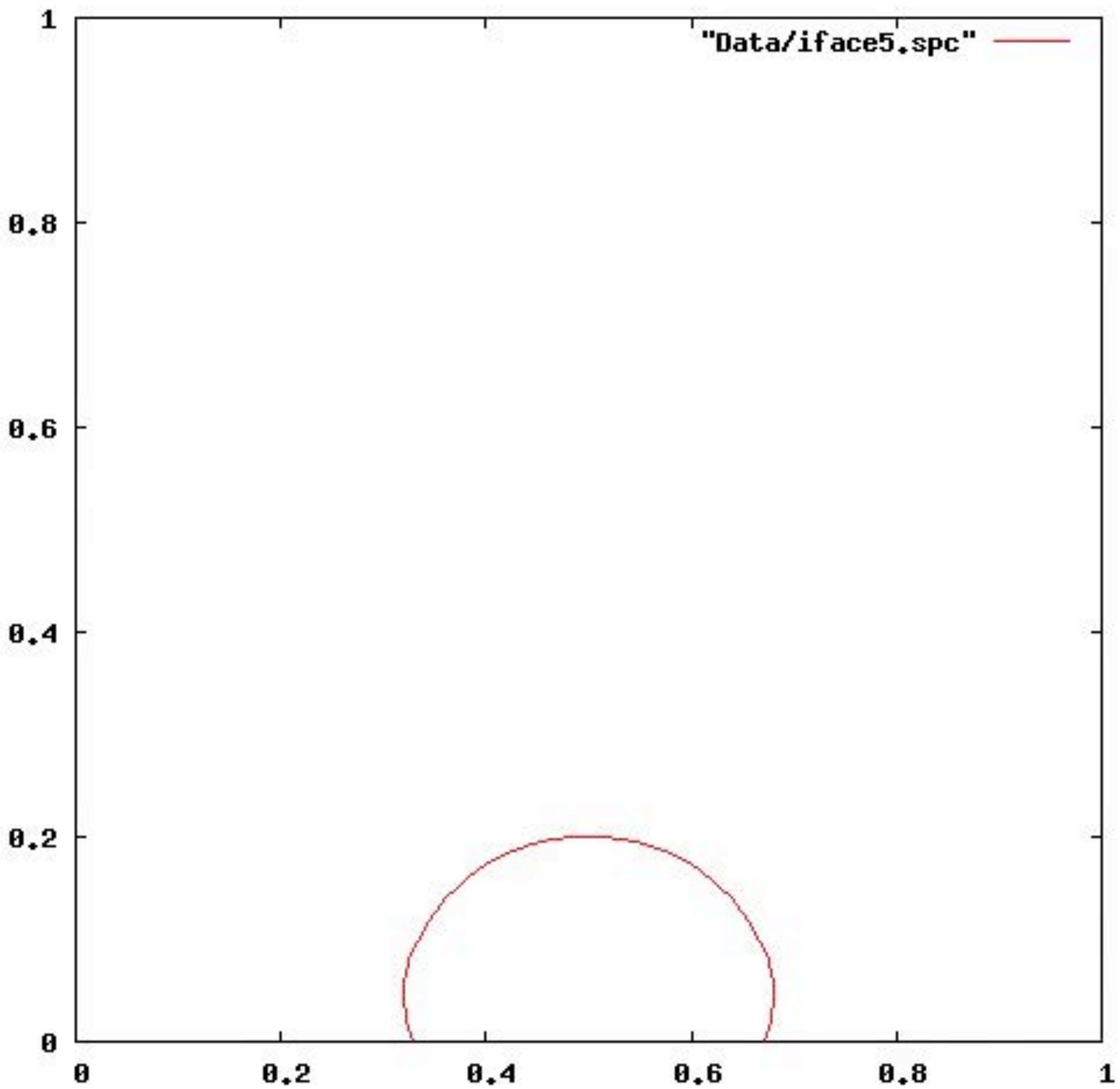
Properties of the approximation scheme

Multiphase Volume Preserving HMCF

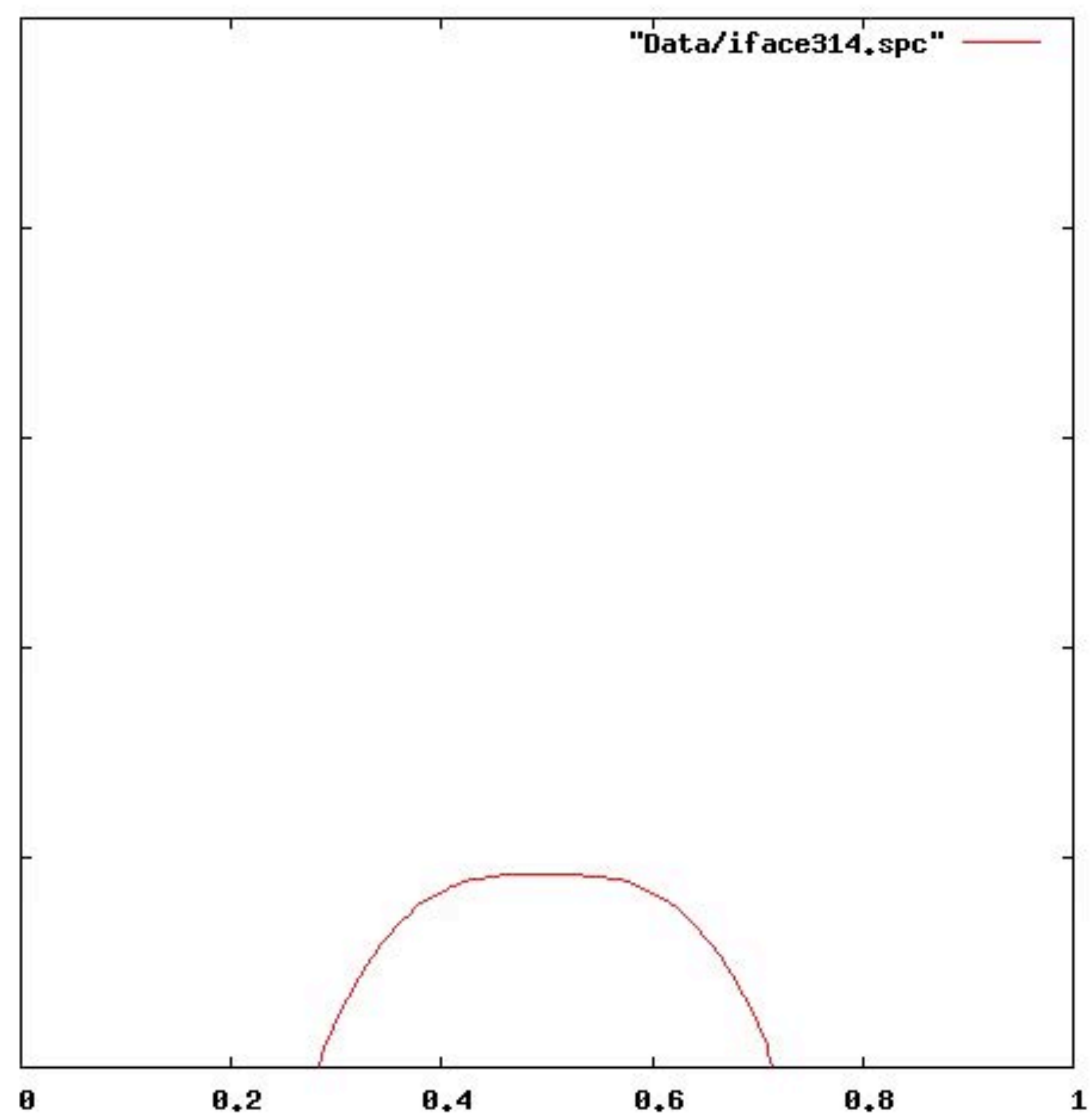


Contact Angles

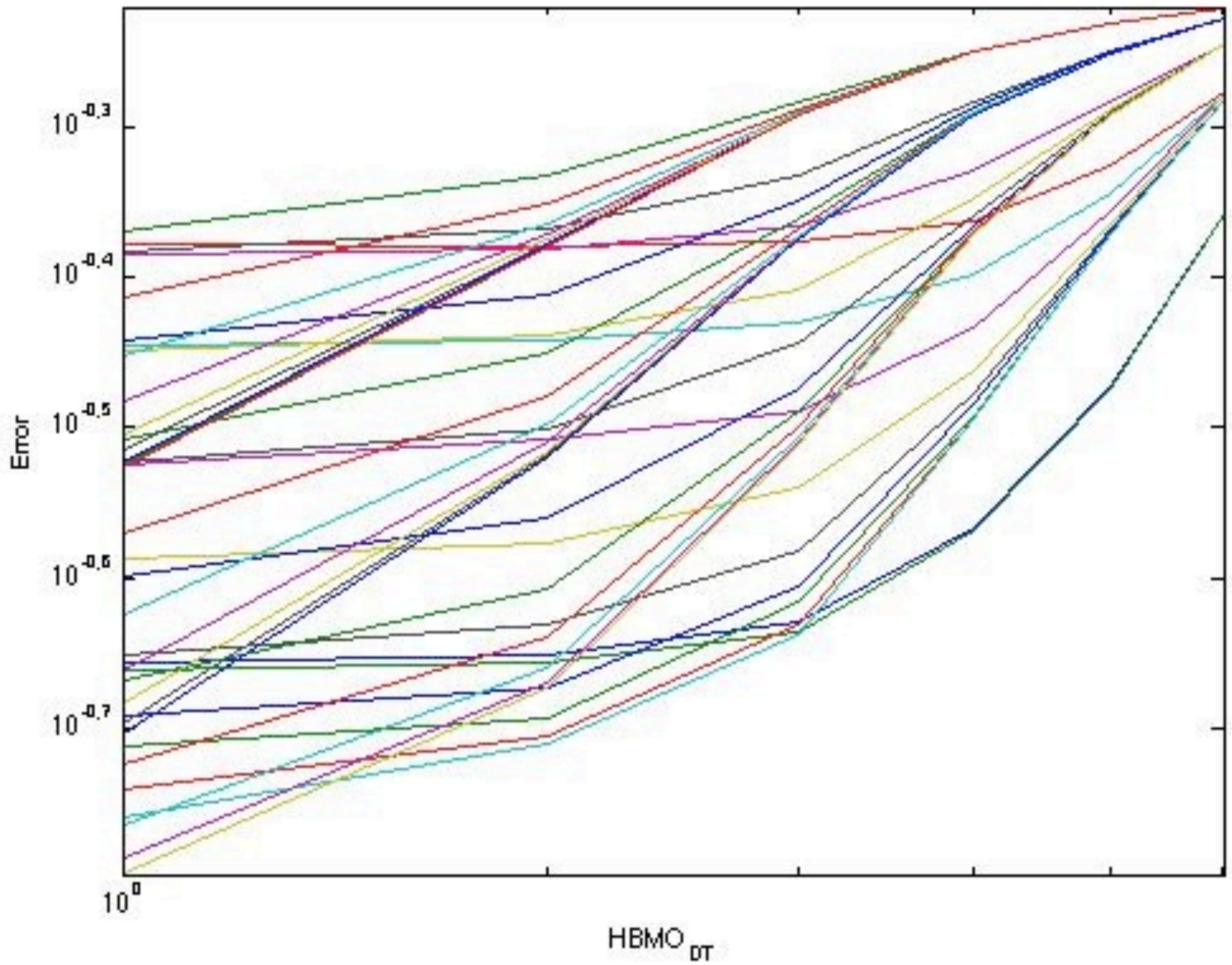
Hydrophobic



Hydrophilic



Error Analysis



Summary

We introduced a method for approximating motion by hyperbolic mean curvature flow (HMCF)

The method is a threshold-dynamical algorithm, of the BMO type

The level set formulation suggested that thresholding evolution by the wave equation should yield the desired dynamics

Using the explicit representation formulas of the wave equation, we showed that the thresholding process yields motion by hyperbolic mean curvature flow, with an error of order t.

Numerical investigations suggest that volume preservation should be possible as well

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1. S. Esedoglu, S. Ruuth, R. Tsai. *Diffusion generated motion using signed distance functions*. J. Comp. Phys. Vol. 229, No. 4 (2010), pp. 1017-1042.

2. P. G. LeFloch, K. Smoczyk. *The hyperbolic mean curvature flow*, Journal de Mathématiques Pures et Appliquées, Vol. 90, No. 6 (2008), pp. 591-614.

Thank you for your attention