The minimal habitat size for spreading in a weak competition system with two free boundaries

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This talk is based on a joint work with Jong-Shenq Guo‡

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Boundary Layers in Reaction-Diffusion Phenomena
Meiji University, Japan
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Talk Outline

I Motivation

Competition, spreading: Lotka-Volterra type competition-diffusion (1D habitat):

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u - kv), & x \in \mathbb{R}, \ t > 0, \\ v_t = d_2 v_{xx} + r_2 v(1 - v - hu), & x \in \mathbb{R}, \ t > 0, \end{cases} \tag{1.1}$$

where all parameters are positive and

- $u(x,t), v(x,t)$: population densities;
- d_1, d_2: diffusion coefficients;
- k, h: competition coefficients;
- r_1, r_2: intrinsic growth rates.

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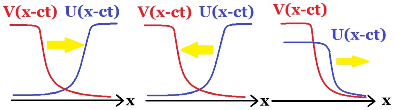
- Tang-Fife (1980), Gardner (1982), Conley-Gardner (1984), Kan-on (1995,1997)...etc



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$z := x - ct$

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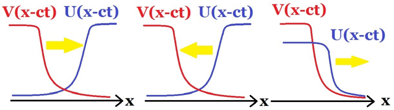

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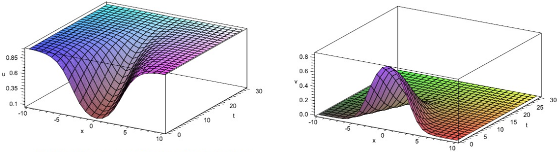
- Front-like entire solutions: Morita-Tachibana (2009)



A bird's eye view of u and v

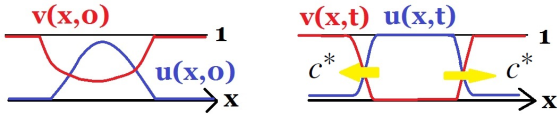
The picture is adapted from *Y. Morita and K.Tachibana An entire solution to the Lotka-Volterra competition-diffusion equations*, SIAM J. Math. Anal 40 (2009), 2217-2240.

I Motivation

- Asymptotic spreading speed:
  - Weinberger-Lewis-Li (2002), Lewis-Li-Weinberger (2005), Li-Weinberger-Lewis (2005), Liang-Zhao (2007,2010)...etc
  - For example, if u is stronger than v,

$$\lim_{t\to\infty} sup\{[1-u(x,t)]^2+v^2(x,t):|x|\leq(c^*-\varepsilon)t\}=0,$$
$$\lim_{t\to\infty} sup\{[1-v(x,t)]^2+u^2(x,t):|x|\geq(c^*+\varepsilon)t\}=0,$$

for any $\varepsilon>0$.



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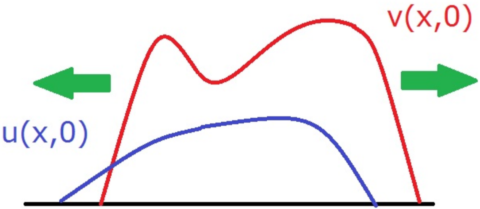
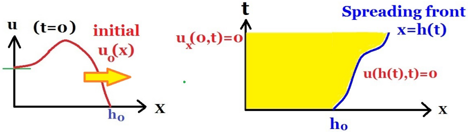


- How to describe the spreading of two species in 1D?

Spreading front as a free boundary

- Du and Lin (2010):

$$\begin{cases} u_t = du_{xx} + u(a - bu), & 0 < x < h(t), \quad t > 0, \\ u_x(0,t) = 0, \quad u(h(t),t) = 0, & t > 0, \\ h'(t) = -\mu u_x(h(t),t), & t > 0, \end{cases}$$



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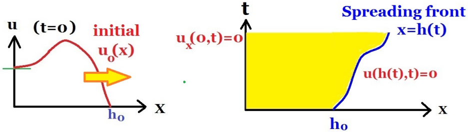

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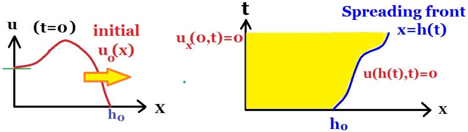

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  - Spreading: $\lim_{t\to+\infty} h(t) := h_\infty = \infty$ and $u \to a/b$ as $t \to \infty$

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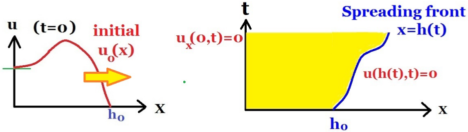

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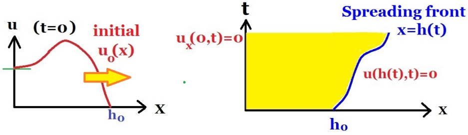

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  - Spreading: $\lim_{t\to+\infty} h(t) := h_\infty = \infty$ and $u \to a/b$ as $t \to \infty$

    or
  - Vanishing: $h_\infty \leq (\pi/2)\sqrt{d/a}$ and $u \to 0$ as $t \to \infty$
- $h(t) = (c_0 + O(1))t$ as $t \to \infty$, c_0 is called the asymptotic spreading speed

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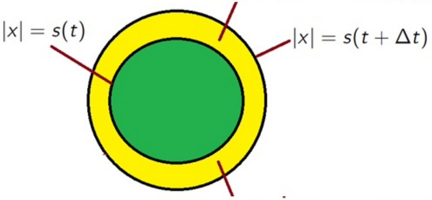
- Bunting, Du and Krakowski (2012 NHM): "population loss" at the spreading front.

Enter: E:= $d_1|\nabla_x u|\Delta t \times$ (surface area of $\{x = s(t)\}$)

$= d_1|\nabla_x u|\Delta t \times N\omega_N[s(t)]^{N-1}$



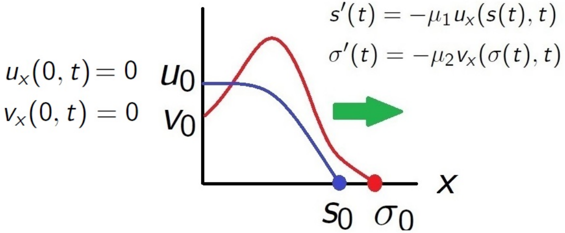
$$lim_{\Delta t \to 0}\frac{E-L}{\Delta t} = 0$$

$\Rightarrow$ $d_1|\nabla_x u| = ks'(t)$

Loss: L:= $k \times$ (volume of the region) $= k[\omega_N(s(t+\Delta t)^N - s(t)^N)]$

II Spreading front as a free boundary: two species case

- Guo and Wu (2014):



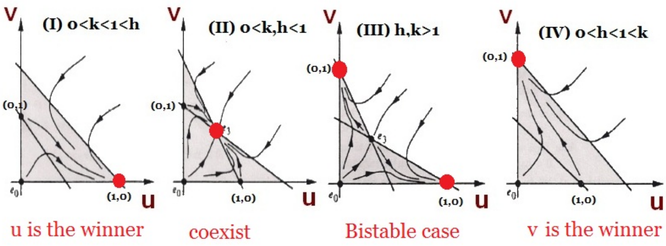
- Two species have their own spreading front.
- Two spreading fronts may intersect each other.

II Spreading front as a free boundary: two species case

- Guo-Wu (2014) study the following problem **(P)** with $0 < k < 1 < h$ (u is superior competitor):

$$
\begin{aligned}
&u_t = d_1 u_{xx} + r_1 u(1 - u - kv), \quad 0 < x < s(t),\ t > 0,\\
&v_t = d_2 v_{xx} + r_2 v(1 - v - hu), \quad 0 < x < \sigma(t),\ t > 0,\\
&u_x(0,t) = v_x(0,t) = 0,\ t > 0,\\
&u \equiv 0 \quad \text{for } x \geq s(t) \text{ and } t > 0;\ v \equiv 0 \quad \text{for } x \geq \sigma(t) \text{ and } t > 0,\\
&s'(t) = -\mu_1 u_x(s(t),t);\ \sigma'(t) = -\mu_2 v_x(\sigma(t),t) \text{ for } t > 0,\\
&(s,\sigma)(0) = (s_0,\sigma_0),\ (u,v)(x,0) = (u_0,v_0)(x) \text{ for } x \in [0,\infty),
\end{aligned}
$$

II Spreading front as a free boundary: two species case



- In ODE sense: u always wipes out v if $0 < k < 1 < h$.
- Guo-Wu (2014): The inferior competitor v can survive if $0 < k < 1 < h$!

II Spreading front as a free boundary: two species case

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- Yes if $0 < h, k < 1$ (Wu, 2014).

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- Yes if $0 < h, k < 1$ (Wu, 2014).
- Given d_i, r_i $(i = 1, 2)$, h and k (the parameters in u and v-equation), there exists $s_{\min}$ in the sense that it is the minimal value such that $s_0 \geq s_{\min}$ guarantees the spreading of u, regardless of σ_0, u_0, v_0 and μ_i, $i = 1, 2$, but it can vanish eventually if $s_0 < s_{\min}$.

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- We do not know if $0 < k < 1 < h$!
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- Given d_i, r_i $(i = 1, 2)$, h and k (the parameters in u and v-equation), there exists $s_{\min}$ in the sense that it is the minimal value such that $s_0 \geq s_{\min}$ guarantees the spreading of u, regardless of σ_0, u_0, v_0 and μ_i, $i = 1, 2$, but it can vanish eventually if $s_0 < s_{\min}$.
- $s_{\min}$: the minimal habitat size for spreading of u.

II Spreading front as a free boundary: two species case

- Hereafter, we always assume **(H)**: $0 < h, k < 1$.
- Let $s_\infty := \lim_{t\to\infty} s(t)$ and $\sigma_\infty := \lim_{t\to\infty} \sigma(t)$
- We introduce the following four quantities:

$$s_* := \frac{\pi}{2}\sqrt{\frac{d_1}{r_1}}, \quad s^* := \frac{\pi}{2}\sqrt{\frac{d_1}{r_1}}\frac{1}{\sqrt{1-k}},$$
$$\sigma_* := \frac{\pi}{2}\sqrt{\frac{d_2}{r_2}}, \quad \sigma^* := \frac{\pi}{2}\sqrt{\frac{d_2}{r_2}}\frac{1}{\sqrt{1-h}}.$$

- $s_* < s^*$ and $\sigma_* < \sigma^*$.

II Spreading front as a free boundary: two species case

- (Vanishing) The species u *vanishes eventually* if $s_\infty < +\infty$ and

$$\lim_{t\to+\infty} \|u(\cdot,t)\|_{C([0,s(t)])} = 0;$$

- (Spreading) The species u *spreads successfully* if $s_\infty = +\infty$ and the species u *persists* in the sense that

$$\liminf_{t\to\infty} u(x,t) > 0$$

uniformly in any bounded interval of $[0,\infty)$.

III Main results

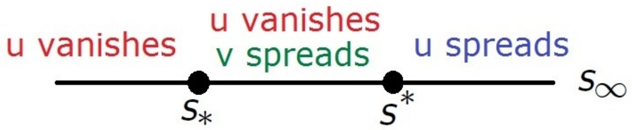
Theorem

Assume **(H)**. *Then the followings hold:*

(i) *If* $s_\infty \le s_*$, *then u vanishes eventually.*

(ii) *If* $s_\infty \in (s_*, s^*]$, *then u vanishes eventually and v spreads successfully.*

(iii) *If* $s_\infty > s^*$, *then u spreads successfully.*



III Main results

Theorem

Assume **(H)**. *Then the followings hold:*

(i) *If* $\sigma_\infty \leq \sigma_*$, *then v vanishes eventually.*

(ii) *If* $\sigma_\infty \in (\sigma_*, \sigma^*]$, *then v vanishes eventually and u spreads successfully.*

(iii) *If* $\sigma_\infty > \sigma^*$, *then v spreads successfully.*

III Main results

Corollary (spreading-vanishing quartering)

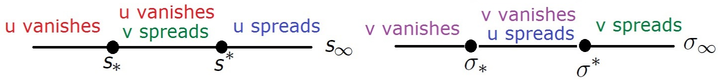
Assume **(H)**. *Then the dynamics of* **(P)** *can be classified into four cases:*

(i) *both two species vanish eventually. In this case,* $s_\infty \leq s_*$ *and* $\sigma_\infty \leq \sigma_*$,

(ii) u *vanishes eventually and* v *spreads successfully. In this case,* $s_\infty \leq s^*$,

(iii) u *spreads successfully and* v *vanishes eventually. In this case,* $\sigma_\infty \leq \sigma^*$.

(iv) *both two species spreading successfully.*



III Main results

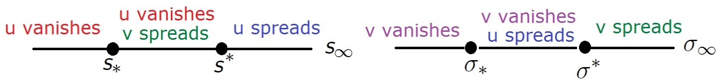
Theorem

Assume **(H)**. *Then the followings hold:*

(a) *If* $s_0 \geq s^*$, *then u spreading successfully, regardless of* u_0, v_0, σ_0.

(b) *If* $s_0 < s_*$ *and* $\|u_0\|_{L^\infty}$ *is small enough, then u vanishes eventually.*

(c) *If* $s_0 < s_*$, $\|u_0\|_{L^\infty}$ *is small enough and* $\sigma_0 \geq \sigma^*$, *then u vanishes eventually and v spreading successfully.*



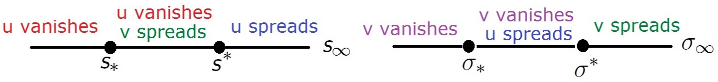
III Main results

Theorem

Assume **(H)**. *Then the followings hold:*

(d) *If* $\sigma_0 < \sigma_*$, $\|v_0\|_{L^\infty}$ *is small enough and* $s_0 \geq s^*$, *then v vanishes eventually and u spreading successfully.*

(e) *Let* $s_0 \in (s_*, s^*)$ *and* $\sigma_0 > \sigma^*$ *(so v spreading successfully). Then the species u vanishes eventually with* $s_\infty \in (s_*, s^*)$ *as long as h and* μ_1 *are small enough.*



III Main results

- Recall that if $s_\infty > s^*$, then u spreads successfully.
- Question: for any given d_i, r_i $(i = 1, 2)$, h and k, the parameters in u-equation and v-equation, does there exist a $s_{\min}$ in the sense that it is the minimal value such that $s_0 \geq s_{\min}$ guarantees the spreading of u, regardless of σ_0, u_0, v_0 and μ_i, $i = 1, 2$, but it can vanish eventually if $s_0 < s_{\min}$
- If such $s_{\min}$ exists, we call it the minimal habitat size for spreading of u.

III Main results

Theorem (The minimal habitat size for spreading of u)

Assume **(H)** *and let d_i, r_i $(i = 1, 2)$, h and k be given. Then there exists minimal habitat size for spreading*

$$s_{\min} := \min\{\hat{s} > 0 |\ u \text{ always spreads successfully if } s_0 = \hat{s}\}$$

such that the species u spreads successfully, regardless of u_0, v_0, σ_0 and the parameters μ_i, $i = 1, 2$ if and only if $s_0 \geq s_{\min}$. Furthermore,

$$\frac{\pi}{2}\sqrt{\frac{d_1}{r_1}\left(\frac{1-hk}{1-k}\right)} \leq s_{\min} \leq s^* \tag{3.2}$$

Ideas of proofs

- Define

$$
\begin{aligned}
A := & \{\hat{s} > 0 |\, u \text{ always spreads successfully if } s_0 = \hat{s}, \\
& \text{regardless of } u_0,\ v_0,\ \sigma_0 \text{ and } \mu_i\}.
\end{aligned}
$$

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- Claim: $s_{\min} \in A$.
- Claim:

$$\frac{\pi}{2}\sqrt{\frac{d_1}{r_1}\left(\frac{1-hk}{1-k}\right)} \leq s_{\min}.$$

Using a contradiction argument.

III Main results

Corollary (The minimal habitat size for spreading of v)

Assume **(H)** *and let d_i, r_i $(i = 1, 2)$, h and k be given. Then there exists minimal habitat size for spreading*

$$\sigma_{\min} := \min\{\hat{\sigma} > 0 |\ v \textit{ always spreads successfully if } \sigma_0 = \hat{\sigma}\}$$

such that the species v spreads successfully, regardless of u_0, v_0, s_0 and the parameters μ_i, $i = 1, 2$ if and only if $\sigma_0 \geq \sigma_{\min}$. Furthermore,

$$\frac{\pi}{2}\sqrt{\frac{d_2}{r_2}\left(\frac{1-hk}{1-h}\right)} \leq \sigma_{\min} \leq \sigma^*$$

III Main results

Theorem (Long-time behavior)

Assume that **(H)** *and* $s_\infty = \sigma_\infty = \infty$. *Then for each* $l \geq 0$,

$$\lim_{t\to\infty} \max_{0\leq x\leq l} \left| u(x,t) - \frac{1-k}{1-hk} \right| = 0,$$

$$\lim_{t\to\infty} \max_{0\leq x\leq l} \left| v(x,t) - \frac{1-h}{1-hk} \right| = 0.$$

- Iteration scheme. Construct some suitable sequences $\{\underline{u}_n\}$, $\bar{u}_n$, $\underline{v}_n$ and $\bar{v}_n$.

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- Iteration scheme. Construct some suitable sequences $\{\underline{u}_n\}$, $\bar{u}_n$, $\underline{v}_n$ and $\bar{v}_n$.
- For each $l \geq 0$,

$$\underline{u}_n \leq \liminf_{t\to+\infty} u(x,t) \leq \limsup_{t\to+\infty} u(x,t) \leq \bar{u}_n,$$

$$\underline{v}_n \leq \liminf_{t\to+\infty} v(x,t) \leq \limsup_{t\to+\infty} v(x,t) \leq \bar{v}_n,$$

IV Discussion

- In the weak competition case, the larger initial habitat size the species owns, the more benefit the species has for spreading.
- If $v \equiv 0$, we have $s_{\min} = s_*$, which is exactly the critical length in single species established by Du-Lin (2010).
- As $h \to 0$, $s_{min} \uparrow s^*$. It means that the species u becomes weaker, it becomes more challenge for successful spreading. Similarly, we have such result for v.

END

Thank you for your attention!