Lecture Note Series
in Mathematical Sciences Based on Modeling and Analysis

ICMMA 2023

MIMS / CMMA International Conference on
"Reaction-diffusion systems: from the past to the future"
— in memory of Prof.Masayasu Mimura —

Meiji Institute for Advanced Study of Mathematical Sciences MIMS
Center for Mathematical Modeling and Applications CMMA

Lecture Note Series

2023 MIMS / CMMA International Conference on
"Reaction-diffusion systems: from the past to the future"
— in memory of Prof.Masayasu Mimura —
(ICMMA 2023)

Meiji University,
1-1 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8301, Japan

Supported by:
Meiji Institute for Advanced Study of Mathematical Sciences (MIMS),
Center for Mathematical Modeling and Applications, Meiji University(CMMA)

Meiji Institute for Advanced Study of Mathematical Sciences (MIMS)
8F High-Rise Wing, Nakano Campus, Meiji University,
4-21-1 Nakano, Nakanoku, Tokyo, Japan, 164-8525


ICMMA 2023
International Conference on
Reaction-diffusion systems:
from the past to the future
— in memory of Prof. Masayasu Mimura —
October 31 - November 2, 2023
Poster presentations will be held on November 1 (on-site only).
On site: Nakano Campus, Meiji University
On-line: Zoom Webinar
MEXT Joint Usage / Research Center "Center for Mathematical Modeling and Applications"(CMMA)
Meiji Institute for Advanced Study of Mathematical Sciences (MIMS)

Preface

The International Conference on "Reaction-diffusion systems: from the past to the future" was held in memory of Professor Masayasu Mimura as International Conference on Mathematical Modeling and Applications (ICMMA) 2023. The conference took place from October 31 to November 2, 2023, at Nakano Campus.

The conference aimed to address the contemporary advancements and forthcoming hurdles in the study of complex pattern dynamics appearing in reaction-diffusion systems. Additionally, it served as a platform to inspire the next generation of researchers by sharing Professor Mimura's profound dedication to this field. Despite being organized as a hybrid event, blending online participation with face-to-face interactions, participants actively fostered the advancement of research in this domain through engaging discussions.

The members of the program committee were Hiroshi Matano, Ken-Ichi Nakamura, Hirokazu Ninomiya and Toshiyuki Ogawa. The conference was organized by Kota Ikeda, Ken-Ichi Nakamura, Hirokazu Ninomiya, Hiraku Nishimori, Masashi Shiraishi, Joe Yuichiro Wakano and Toshiyuki Ogawa (chair).

The papers assembled for this volume are based on the presentation slides of the speakers. We are grateful to the MIMS staff for their support.

February 23, 2024
Toshiyuki Ogawa

ICMMA 2023

International Conference on Reaction-diffusion systems: from the past to the future
— in memory of Prof. Masayasu Mimura —

October 31, 2023

9:40 ~ 9:50	●	Opening
9:50 ~10:30	●	Arnaud Ducrot (Université Le Havre Normandie, France) "Spatial propagation for nonlocal non-autonomous Fisher-KPP equation"
10:40 ~11:20	●	Joe Yuichiro Wakano (Meiji University, Japan) "Ecocultural range-expansion model of modern humans in Paleolithic"
11:20	●	Coffee Break
11:50 ~12:30	●	Jong-Shenq Guo (Tamkang University, Taiwan) "Traveling wave solutions for a three-species diffusive competition system"
12:30	●	Break
14:00 ~14:40	●	Hideo Ikeda (University of Toyama, Japan) "Stability of single transition layer solutions in mass-conserving reaction-diffusion systems with bistable nonlinearity"
14:50 ~15:30	●	Hirofumi Izuhara (Miyazaki University, Japan) "Traveling wave solutions of combustion in a narrow channel"
15:30	●	Coffee Break
16:20~17:00	●	Danielle Hilhorst (Université Paris-Saclay, France) "Two phase Stefan problems as the singular limit of competition-diffusion systems arising in population dynamics"

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November 1, 2023

9:50 ~10:30	Yasumasa Nishiura (Hokkaido University, Japan) "The floodgates to pattern formation problems"
10:40 ~11:20	Yoshitaro Tanaka (Future University Hakodate, Japan) "Keller-Segel type approximation for nonlocal Fokker-Planck equations in one-dimensional bounded domain"
11:20	Coffee Break
11:50 ~12:30	Chang-Hong Wu (National Yang Ming Chiao Tung University, Taiwan) "Spreading fronts arising from the singular limit of reaction-diffusion systems"
12:30	Break
14:00 ~16:00	Poster Session (on-site only)
16:00	Coffee Break
17:00~19:00	Banquet / Poster Awards

ICMMA 2023

International Conference on Reaction-diffusion systems: from the past to the future
— in memory of Prof. Masayasu Mimura —

November 2, 2023

9:50 ~10:30	Yoshihisa Morita (Ryukoku University, Japan) "Segregation pattern in a reaction-diffusion model of asymmetric cell division"
10:40 ~11:20	Quentin Griette (Université Le Havre Normandie, France) "Speed-up of traveling waves by negative chemotaxis"
11:20	Coffee Break
11:50 ~12:30	Shin-Ichiro Ei (Hokkaido University, Japan) "A Billiard Problem in Nonlinear Dissipative Systems"
12:30	Break
14:00 ~14:40	Philippe Souplet（Université Sorbonne Paris Nord, France） "Convergence, concentration and critical mass phenomena for a model of cell migration with signal production on the boundary"
14:50 ~15:30	Hiroshi Matano (Meiji University, Japan) "Front propagation in the presence of obstacles"
15:30 ~15:40	Closing

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"Spatial propagation for nonlocal non-autonomous Fisher-KPP equation"

Arnaud Ducrot (Université Le Havre Normandie, France)

In this lecture we discuss existence results of travelling wave solutions and spreading speed for a non-autonomous Fisher-KPP equation with nonlocal diffusion. We prove that under suitable time averaging properties for the coefficients, the equation exhibits a definite spreading speed. We also study non-autonomous Fisher-KPP equation on a lattice and deduce from our analysis some spreading phenomenon for some predator-prey systems on lattice.

Spatial propagation for nonlocal non-autonomous Fisher-KPP equation

Arnaud Ducrot

Université Le Havre Normandie
UR 3821 LMAH
Joint works with Zhucheng Jin

ICMAA 2023
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Meiji University, MIMS.

October 31, 2023

Discrete diffusion

On the lattice $\mathbb{Z}$, individuals can jump from j to i with probability $p(j,i)$. Hence the density of population at i is given by

$$\frac{\partial u(t,i)}{\partial t} = \underbrace{\sum_{j\in\mathbb{Z}} p(j,i)u(t,j)}_{\text{Coming at } i \text{ from any } j} - \underbrace{\sum_{j\in\mathbb{Z}} p(i,j)u(t,i)}_{\text{leaving } i \text{ to somewhere}}$$

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If $p(j,i) = J(i-j)$ depends on the distance between the point j and i, we get:

$$\frac{\partial u(t,i)}{\partial t} = \underbrace{\sum_{j\in\mathbb{Z}} J(i-j)u(t,j)}_{\text{discrete convolution}} - \left(\sum_{j\in\mathbb{Z}} J(j-i)\right) u(t,i)$$

Continuous version

The discrete lattice is changed to the line and the probability transitions are given with the kernel $J(x-y)$ from y to x.

Non-local heat equation

$$\frac{\partial u(t,x)}{\partial t} = \int_{\mathbb{R}} J(x-y)u(t,y)dy - u(t,x)\int_{\mathbb{R}} J(y-x)dy$$

or equivalently,

$$\frac{\partial u(t,x)}{\partial t} = \int_{\mathbb{R}} J(x-y)\big(u(t,y)-u(t,x)\big)dy.$$

Non-local heat kernel

If J has a unit mass, the fundamental solution $K(t, x - x')$ is given by:

- Using Fourier transform:

$$\widehat{K(t,\cdot)}(\xi) = e^{t(\widehat{J}(\xi)-1)}.$$

- Using exponential $e^{tJ*\cdot}$:

$$K(t,x) = e^{-t}\delta_0(dx) + \Psi(t,x) \text{ with } \Psi(t,x) = e^{-t}\sum_{k=1}^{\infty}\frac{t}{k!}J^{*k}(x)$$

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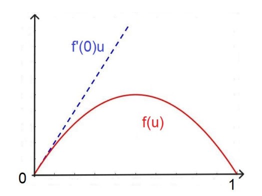
The solution with initial u_0 reads as

$$u(t,x) = e^{-t}u_0(x) + \text{rather smooth fonction.}$$

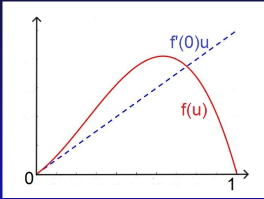
Fisher-KPP equation with nonlocal diffusion

$$\partial_t u(t,x) = \int_{\mathbb{R}} J(x-y)\left[u(t,y) - u(t,x)\right] dy + f(u(t,x)), \quad t > 0, x \in \mathbb{R},$$

where $f(0) = f(1) = 0$, f is **KPP type**, that is:
f is smooth, $f(0) = f(1) = 0$ and $0 < f(u) \leq f'(0)u$ for $u \in (0,1)$.
Typical example: $f(u) = u(1-u)$.



KPP



Monostable

Travelling wave solutions

Travelling waves: Special solution of the form

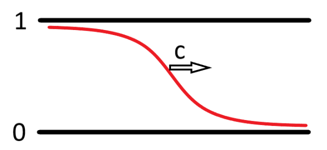
$$u(t,x) = \phi(x - ct)$$

which connects the stationary solutions 0 and 1

ϕ is the profile and c is the wave speed

Wave profile equation: $\phi = \phi(z)$ satisfies

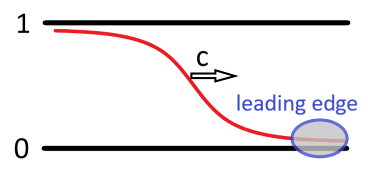
$$\begin{cases} \int_{\mathbb{R}} J(y)\left[\phi(z-y) - \phi(z)\right] dy + c\phi'(z) + f(\phi(z)) = 0, \; z \in \mathbb{R} \\ \phi(-\infty) = 1 \text{ and } \phi(+\infty) = 0. \end{cases}$$



Travelling wave solutions for thin tailed kernel

- Wave profile ϕ satisfies

$$\begin{cases} \int_{\mathbb{R}} J(y)\left[\phi(z-y)-\phi(z)\right] dy + c\phi'(z) + f(\phi(z)) = 0, \\ \phi(-\infty) = 1 \text{ and } \phi(+\infty) = 0. \end{cases}$$



- At the leading edge $\phi \approx 0$ and $\int_{\mathbb{R}} K(y)\left[\phi(z-y)-\phi(z)\right] dy + c\phi'(z) + f'(0)\phi(z) \approx 0.$

Travelling wave solutions for thin tailed kernel

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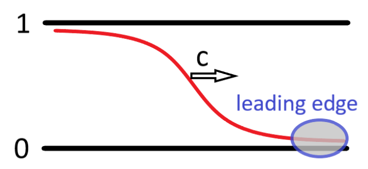

- At the leading edge $\phi \approx 0$ and
  $\int_{\mathbb{R}} K(y)\left[\phi(z-y)-\phi(z)\right] dy + c\phi'(z) + f'(0)\phi(z) \approx 0.$
  For $\phi(z)=e^{-\lambda z}$, with $\lambda > 0$ we get:

$$\int_{\mathbb{R}} J(y)[e^{\lambda y}-1]dy - \lambda c + f'(0) = 0.$$

Known results: travelling waves

Theorem (Schumacher 1980; Carr, Chmaj 2004;...)

If f is of **KPP type** *and J is a* ***thin-tailed kernel****, that is $\exists \beta > 0,\ \int_{\mathbb{R}} e^{\beta |y|} J(y)dy < \infty$, there exists a travelling wave solution with speed c if and only if $c \geq c^*$. The travelling wave solution is unique up to translation. The minimal wave speed c^* is characterized by*

$$c^* := \inf_{\lambda>0} \lambda^{-1} \left(\int_{\mathbb{R}} J(y)[e^{\lambda y} - 1]dy + f'(0) \right).$$

- Monostable type: Coville, Dupaigne 2007; Liang, Zhao 2007; Yagisita 2010; Fang, Zhao 2014;...
- Bistable or Ignition type: Bates, Fife, Ren, Wang 1997; Chen 1997; Coville, Dupaigne 2005;...

Known results: spreading speeds

$$\partial_t u = J * u - u + f(u), \quad t > 0, x \in \mathbb{R}, \;\; u(0,x) = u_0(x).$$

Theorem (Lutscher, Pachepsky, Lewis 2005; Liang, Zhao 2007;...)

Assume that J is thin-tailed and symmetric. If the initial data u_0 is compactly supported, then there exists $c^{**} > 0$ such that

$$\begin{cases} \lim_{t\to\infty} \sup_{|x|\geq ct} u(t,x) = 0, & \forall c > c^{**}, \\ \lim_{t\to\infty} \sup_{|x|\leq ct} |1 - u(t,x)| = 0, & \forall c \in [0, c^{**}). \end{cases}$$

*c^{**} is called the **spreading speed** and we have $c^{**} = c^*$.*

Remark: Non-symmetric kernel J: Xu, Li, Ruan 2021.

Our goal

Study the propagation for the non-autonomous problem

$$\partial_t u(t,x) = \int_{\mathbb{R}} K(t,x-y)\,[u(t,y)-u(t,x)]\,dy + f(t,u(t,x)).$$

- Generalized travelling waves
- Spreading speed
- discrete convolution: Fisher-KPP on lattice
- Application for predator-prey system on lattice.

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- Generalized travelling waves
- Spreading speed
- discrete convolution: Fisher-KPP on lattice
- Application for predator-prey system on lattice.

for simplicity we choose $f(t,u) = \mu(t)u(1-u)$.

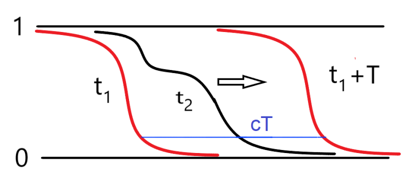
Time periodic coefficients

$$\partial_t u(t,x) = \int_{\mathbb{R}} K(x-y)\,[u(t,y)-u(t,x)]\,dy + f(t,u(t,x)).$$

$\exists T>0\ f(t+T,u)\equiv f(t,u)$ of **KPP type** and thin-tailed kernel.

• **Pulsating wave** [Shigesada, Kawasaki, Teramoto 1986; Alikakos, Bate, Chen 1999]:

$$u(t,x)=\phi(t,x-ct),\quad \phi(t,\cdot)=\phi(t+T,\cdot),$$
$$\phi(t,-\infty)=1 \text{ and } \phi(t,+\infty)=0,\ \text{ unif. for } t\in\mathbb{R}.$$

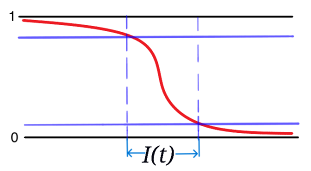


More general time heterogeneities

Definition (Berestycki, Hamel 2007, 2012; Matano 2003; Shen 2004)

A generalized transition front connecting 0 and 1 is an entire solution $u = u(t,x)$ and an interface function $X : \mathbb{R} \to \mathbb{R}$ s.t.

$$u(t, x + X(t)) \to \begin{cases} 1, & \text{as } x \to -\infty, \\ 0, & \text{as } x \to +\infty; \end{cases} \quad \text{unif. for } t \in \mathbb{R}.$$



More general time heterogeneities

- Existence of generalized transition front [Shen, Shen 2016]

$$\partial_t u(t,x) = \int_{\mathbb{R}} K(x-y)\,[u(t,y)-u(t,x)]\;dy + f(t,u(t,x)),$$

where f is of **KPP type** and K is a **thin-tailed** kernel.

More general time heterogeneities

Definition (Berestycki, Hamel 2007, 2012; Matano 2003; Shen 2004)

A generalized transition front connecting 0 and 1 with interface X is a solution $u = u(t,x)$

$$\lim_{x\to-\infty} u(t,x+X(t)) = 1 \text{ and } \lim_{x\to+\infty} u(t,x+X(t)) = 0, \text{ unif. for } t\in\mathbb{R}.$$

Definition (Nadin, Rossi 2012; Shen 2011)

A generalized travelling wave solution u with speed function $c = c(t) \in L^\infty(\mathbb{R})$ is nothing but a **generalized transition front** with a globally Lipschitz continuous interface function

$$X(t) = \int_0^t c(s)\,ds.$$

Least mean value and mean value

- **Least mean value** [Nadin, Rossi 2012]: for $g \in L^\infty(\mathbb{R})$,

$$\lfloor g \rfloor := \lim_{T \to +\infty} \inf_{t \in \mathbb{R}} \frac{1}{T} \int_0^T g(t+s)\, ds.$$

Least mean value and mean value

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- **Mean value:** for $g \in L^\infty(\mathbb{R})$,

$$\langle g \rangle := \lim_{T \to +\infty} \frac{1}{T} \int_0^T g(t+s) ds, \text{ unif. for } t \in \mathbb{R}.$$

Remark:

1. Some typical classes of heterogeneities admit a **mean value.**
(e.g. **periodic**, **almost periodic**, so on...)
2. If $g \in L^\infty(\mathbb{R})$ admits a **mean value**, then $\langle g \rangle = \lfloor g \rfloor$.

The problem

$$\partial_t u = \int_{\mathbb{R}} K(t, x-y)\left[u(t,y) - u(t,x)\right] dy + \mu(t) u\left(1-u\right), (t,x) \in \mathbb{R}^2.$$

- **Aim:** existence of generalized travelling wave solution with speed $c = c(t) \in L^\infty(\mathbb{R})$.

The problem

$$\partial_t u = \int_{\mathbb{R}} K(t, x-y)\,[u(t,y) - u(t,x)]\,dy + \mu(t)u\,(1-u)\,, (t,x) \in \mathbb{R}^2.$$

- **Aim:** existence of generalized travelling wave solution with speed $c = c(t) \in L^\infty(\mathbb{R})$.
- Wave profile $\phi(t,z) := u\left(t, z + \int_0^t c(s)\,ds\right)$ (if smooth enough) satisfies

$$\partial_t \phi - c(t)\partial_z \phi = \int_{\mathbb{R}} K(t,y)[\phi(t,z-y) - \phi(t,z)]\,dy + \mu(t)\phi(1-\phi),$$

with

$$\phi(t,-\infty) = 1 \text{ and } \phi(t,+\infty) = 0, \quad \text{unif. for } t \in \mathbb{R}.$$

Assumptions

Assumption:

$$\mu \in L^\infty(\mathbb{R}) \text{ and } \inf_{t\in\mathbb{R}} \mu(t) > 0.$$

Assumption:

(i) K is nonnegative and $K \in L^1_y(\mathbb{R}, L^\infty_t(\mathbb{R}))$.

(ii) (Thin-tailed) For some $\lambda > 0$,

$$\int_{\mathbb{R}} \|K(\cdot,y)\|_\infty e^{\lambda y} dy < \infty.$$

(iii) $\limsup_{\lambda\to abs(K)^-} \frac{\int_{\mathbb{R}} \|K(\cdot,y)\|_\infty e^{\lambda y} dy}{\lambda} = \infty$, where

$abs(K) := \sup\left\{\lambda \geq 0 : \int_{\mathbb{R}} \|K(\cdot,y)\|_\infty e^{\lambda y}\, dy < \infty\right\}.$

Linear determination of speed function

- Linearized equation

$$\partial_t \psi - c(t)\partial_z \psi - \int_{\mathbb{R}} K(t,y)[\psi(t,z-y) - \psi(t,z)]\,dy - \mu(t)\psi = 0.$$

- Ansatz

$$\psi(z) := e^{-\lambda z},\ \ \lambda \in (0, abs(K)).$$

- Derive

$$c(t) = c(\lambda)(t) := \frac{\int_{\mathbb{R}} K(t,y)[e^{\lambda y} - 1]\,dy + \mu(t)}{\lambda}, \lambda \in (0, abs(K)).$$

Linear determination of speed function

- Linearized equation

$$\partial_t\psi - c(t)\partial_z\psi - \int_{\mathbb{R}} K(t,y)[\psi(t,z-y)-\psi(t,z)]\,dy - \mu(t)\psi = 0.$$

- Ansatz

$$\psi(z) := e^{-\lambda z},\ \ \lambda\in(0,abs(K)).$$

- Derive

$$c(t) = c(\lambda)(t) := \frac{\int_{\mathbb{R}} K(t,y)[e^{\lambda y}-1]\,dy + \mu(t)}{\lambda}, \lambda\in(0,abs(K)).$$

Lemma (Decreasing property)

There exists $\lambda^*\in(0,abs(K))$ s.t.

$$\{\lambda>0 : \exists\lambda'>\lambda, \forall k\in(\lambda,\lambda'], \lfloor c(\lambda)-c(k)\rfloor>0\} = (0,\lambda^*).$$

Moreover, $\lambda\mapsto\lfloor c(\lambda)\rfloor$ is decreasing in $(0,\lambda^*)$.

Results

Theorem (Existence)

For each $\lambda \in (0, \lambda^*)$, there exists a generalized travelling wave solution with speed function $c(\lambda) \in L^\infty(\mathbb{R})$.

Remark:

$(\lfloor c(\lambda^*) \rfloor, \infty) \subset \{$least mean value of admissible speed$\}$

Results

Theorem (Existence)

For each $\lambda \in (0, \lambda^*)$, there exists a generalized travelling wave solution with speed function $c(\lambda) \in L^\infty(\mathbb{R})$.

Remark:
$(\lfloor c(\lambda^*) \rfloor, \infty) \subset \{$least mean value of admissible speed$\}$

Theorem (Nonexistence)

Set $\underline{c}(\lambda)(t) := \int_{\mathbb{R}} K(t,y)e^{\lambda y} y\, dy$. There is no generalized travelling wave solution with speed function $c \in L^\infty(\mathbb{R})$ satisfying

$$\lfloor c \rfloor < \lfloor \underline{c}(\lambda^*) \rfloor.$$

Minimal wave speed

Corollary

If $K(\cdot, y)$ for each $y \in \mathbb{R}$ and $\mu(\cdot)$ admit a mean value, then we get

$$c^* := \lfloor c(\lambda^*) \rfloor = \lfloor \underline{c}(\lambda^*) \rfloor$$

and

$\{$*least mean value of admissible speed*$\}$ = *either* (c^*, ∞) *or* $[c^*, \infty)$.

Open question for the mean value c^*.

Sketch of proof of existence theorem

- For $\lambda \in (0, \lambda^*)$,

Super-solution: $\bar{\phi}(t,z) := \min\left\{1, e^{-\lambda z}\right\}$,

Sub-solution: $\underline{\phi}(t,z) := \max\left\{0, e^{-\lambda z} - e^{b(t)} e^{-(\lambda+h)z}\right\}$,

where $h > 0$ and $b \in W^{1,\infty}(\mathbb{R})$.

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  where $h > 0$ and $b \in W^{1,\infty}(\mathbb{R})$.

- The solution $u^n(t,x)$ of **Cauchy problem**

$$\begin{cases} \partial_t u(t,x) = \int_{\mathbb{R}} K(t,y)\left[u(t,x-y) - u(t,x)\right] dy + \mu(t) u(1-u), \\ u(-n,x) = \overline{\phi}(-n, x - \int_0^{-n} c(\lambda)(s)\, ds). \end{cases}$$

Sketch of proof of existence theorem

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  Super-solution: $\bar{\phi}(t,z) := \min\left\{1, e^{-\lambda z}\right\}$,

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- Regularity estimates for $\{u^n\}_{n\geq 1}$: Lipschitz estimates obtained by the comparison principle.
- passing to the limit $n \to \infty$

The problem

Aim: Study the spreading speed for the problem:

$$\begin{cases} \partial_t u = \int_{\mathbb{R}} K(y)\left[u(t,x-y) - u(t,x)\right] dy + \mu(t) u\left(1-u\right), & t>0, x\in\mathbb{R} \\ u(0,x) = u_0(x), \quad 0 \le u_0 \le 1, u_0 \not\equiv 0, & \text{and compact support.} \end{cases}$$

The problem

Aim: Study the spreading speed for the problem:

$$\begin{cases} \partial_t u = \int_{\mathbb{R}} K(y)\left[u(t,x-y) - u(t,x)\right] dy + \mu(t)u\left(1-u\right), \quad t>0, x \in \mathbb{R} \\ u(0,x) = u_0(x), \qquad 0 \leq u_0 \leq 1, u_0 \not\equiv 0, \text{ and compact support.} \end{cases}$$

Assumption: The function μ is uniformly continuous, bounded, and $\inf_{t \geq 0} \mu(t) > 0$.

The problem

Aim: Study the spreading speed for the problem:

$$\begin{cases} \partial_t u = \int_{\mathbb{R}} K(y)\left[u(t,x-y)-u(t,x)\right]dy + \mu(t)u\left(1-u\right), & t>0, x\in\mathbb{R} \\ u(0,x)=u_0(x), \quad 0\le u_0\le 1, u_0\not\equiv 0, & \text{and compact support.} \end{cases}$$

Assumption: The function μ is uniformly continuous, bounded, and $\inf_{t\ge 0}\mu(t)>0$.

Assumption:
The kernel $K:\mathbb{R}\to[0,\infty)$ is continuous, integrable and

(i) $\exists\alpha>0$ s.t. $\int_{\mathbb{R}} K(y)e^{\alpha y}dy<\infty$.

(ii) $K(0)>0$.

(iii) $\lfloor\mu\rfloor>\overline{K}:=\int_{\mathbb{R}} K(y)dy$.

(iv) $\limsup_{\lambda\to abs(K)^-}\frac{\int_{\mathbb{R}} K(y)e^{\lambda y}dy}{\lambda}=\infty$.

Linear determination of speed function

- Similarly, introduce

$$c(\lambda)(t) := \frac{\int_{\mathbb{R}} K(y)e^{\lambda y}dy - \overline{K} + \mu(t)}{\lambda}, \quad \lambda \in (0, abs(K)).$$

- There exists $\lambda^* \in (0, abs(K))$ s.t.

$$\lfloor c(\lambda^*) \rfloor = \inf_{\lambda \in (0, abs(K))} \lfloor c(\lambda) \rfloor.$$

- $\lfloor \mu \rfloor > \overline{K} \Longrightarrow \lfloor c(\lambda^*) \rfloor > 0$.

Main results

Theorem

Let u be a solution with a compactly supported initial data $0 \leq u_0 \leq 1$. Then we have

$$(i)\ \lim_{t\to\infty} \sup_{x\geq\int_0^t c(\lambda^*)(s)ds+\sigma t} u(t,x) = 0,\ \forall\sigma > 0,$$

$$(ii)\ \lim_{t\to\infty} \sup_{x\in[0,ct]} |1-u(t,x)| = 0,\ \forall c \in [0, \lfloor c(\lambda^*)\rfloor).$$

Remark:

- If the coefficients admit a mean value, then one has $\frac{1}{t}\int_0^t c(\lambda^*)(s)ds \to \lfloor c(\lambda^*)\rfloor$ as $t \to \infty$. We obtain the exact spreading speed $\lfloor c(\lambda^*)\rfloor$.

Main results

Theorem (Slower exponential decay initial data)

Let $u(t,x)$ *be the solution with an initial data* u_0*,* $0 \leq u_0 \leq 1$*,* $u_0 \not\equiv 0$ *and such that* $u_0(x) \sim Ce^{-\lambda x}$ *as* $x \to \infty$ *for* $\lambda \in (0, \lambda^*)$*. Then*

$$(i) \lim_{t\to\infty} \sup_{x \geq \int_0^t c(\lambda)(s)ds + \sigma t} u(t,x) = 0, \ \forall \sigma > 0;$$

$$(ii) \lim_{t\to\infty} \sup_{x\in[0,ct]} |1 - u(t,x)| = 0, \ \forall 0 < c < \lfloor c(\lambda) \rfloor.$$

Remark: symmetrical argument yields the spreading properties to the left.
Different decay rates can be considered in the two sides:
$\to$ different speeds of propagation.

Lower wave speed estimate: a key persistence lemma

Lemma

Let $u \in BUC([0,\infty)\times\mathbb{R})$ *be the solution. Define the limit set* $\omega(u)$ *by* $\tilde{u} \in \omega(u)$ *if we have* $\{(t_n, x_n)\}$ *with* $t_n \to \infty$ *and*

$$u(t+t_n, x+x_n) \to \tilde{u}(t,x), \text{ loc. unif. for } (t,x) \in \mathbb{R}^2, \text{ as } n \to \infty.$$

Let $X = X(t) : [0,\infty) \to \mathbb{R}^+$ *continuous and assume that*

(H1) $\liminf_{t\to\infty} u(t,0) > 0;$

(H2) $\exists \varepsilon > 0$ *s.t.* $\liminf_{t\to\infty} \tilde{u}(t,0) \geq \varepsilon, \ \forall \tilde{u} \in \omega(u) \setminus \{0\};$

(H3) $\liminf_{t\to\infty} u(t, X(t)) > 0.$

Then for any $k \in (0,1)$*, one has*

$$\liminf_{t\to\infty} \inf_{0\leq x\leq kX(t)} u(t,x) > 0.$$

Remarks on the persistence lemma

- Examples of propagating path $X(t)$: $X(t) = ct$, $X(t) = \int_0^t c(s)ds$,...
- Without the thin-tailed kernel assumption.
- The proof of the persistence lemma is inspired by Ducrot, Giletti, Matano (CVPDE 2019); Ducrot, Giletti, Guo, Shimojo (Nonlinearity 2021).

The problem

We consider the problem for $t \geq 0$, $i \in \mathbb{Z}$:

$$\frac{d}{dt}w(t,i) = \sum_{j\in\mathbb{Z}} J(t,j)[w(t,i-j)-w(t,i)]+\mu(t)w(t,i)(1-w(t,i)),$$

with an initial data $w(0,i) = w_0(i)$ compactly supported.

We assume that:

• Assumptions for the non-negative kernel: $J \in l^1(\mathbb{Z}, L^\infty(0,\infty))$ is symmetric, thin-tailed, $\inf_{t\geq 0} J(t,\pm 1) > 0$. It has a mean value and

$$\lambda^{-1}\sum_{i\in\mathbb{Z}}\langle J(\cdot,i)\rangle e^{\lambda i} \to \infty \text{ as } \lambda \to \mathrm{abs}(J).$$

• Assumption for $t \mapsto \mu(t)$: uniformly continuous and has a mean value.

Linear speed function

Approximation $w(t,i) \approx e^{-\lambda(i-X(t))}$ with $\lambda > 0$ yields

$$\lambda X'(t) = \sum_{j\in\mathbb{Z}} J(t,j)[e^{\lambda j} - 1] + \mu(t),$$

Define

$$c_\lambda(t) = \lambda^{-1} \sum_{j\in\mathbb{Z}} J(t,j)[e^{\lambda j} - 1] + \lambda^{-1}\mu(t)$$

and its mean value

$$\bar{c}_\lambda = \lambda^{-1} \sum_{j\in\mathbb{Z}} \langle J(.,j)\rangle[e^{\lambda j} - 1] + \lambda^{-1}\langle\mu\rangle.$$

Then there exists $\lambda^* \in (0, \operatorname{abs}(J))$ such that

$$\lambda \mapsto \bar{c}_\lambda \text{ is } \begin{cases} \text{decreasing on } (0,\lambda^*) \\ \text{increasing on } (\lambda^*, \operatorname{abs}(J)) \end{cases}$$

Spreading result

Theorem

If the initial data w_0 is compactly supported then the solution satisfies

$$\lim_{t\to\infty} \sup_{|i|\geq ct} w(t,i) = 0 \text{ for all } c > c^* := \bar{c}_{\lambda^*},$$

and

$$\lim_{t\to\infty} \sup_{|i|\leq ct} |1 - w(t,i)| = 0 \text{ for } 0 < c < c^*.$$

Extend previous results Weinberger 1982; Liang and Zhao 2007; Fang, Wei, Zhao 2010; Cao and Shen 2017; Liang and Zhou 2020, ...

Idea for the proof of the lower bound

- For the lower bound of the speed, we construct a suitable sub-solution with small speed.

- Then coupling with dynamical system arguments and limiting arguments. For limiting arguments, we have to take care of all possible limit equations obtained by time shift $t + t_n$ with $t_n \to \infty$.

Predator-prey systems

Aim: Study the spreading speeds for the prey-predator system for $t > 0$, $i \in \mathbb{Z}$

$$\begin{cases} \frac{d}{dt}u(t,i) = \sum_{j\in\mathbb{Z}} J_1(t,j)[u(t,i-j) - u(t,i)] \\ \qquad +r(t)u(t,i)\big(1-u(t,i)\big) - p(t)u(t,i)v(t,i), \\ \frac{d}{dt}v(t,i) = \sum_{j\in\mathbb{Z}} J_2(t,j)[v(t,i-j) - v(t,i)] \\ \qquad +q(t)u(t,i)v(t,i) - \nu(t)v(t,i), \\ u(0,i) = u_0(i) \text{ and } v(0,i) = v_0(i), \qquad i \in \mathbb{Z}. \end{cases}$$

- **Prey:** $u = u(t,i)$ and **Predator:** $v = v(t,i)$.

Predator-prey systems

Aim: Study the spreading speeds for the prey-predator system for $t > 0$, $i \in \mathbb{Z}$

$$
\begin{cases}
\frac{d}{dt}u(t,i) = \sum_{j\in\mathbb{Z}} J_1(t,j)[u(t,i-j) - u(t,i)] \\
\qquad + r(t)u(t,i)\big(1 - u(t,i)\big) - p(t)u(t,i)v(t,i), \\
\frac{d}{dt}v(t,i) = \sum_{j\in\mathbb{Z}} J_2(t,j)[v(t,i-j) - v(t,i)] \\
\qquad + q(t)u(t,i)v(t,i) - \nu(t)v(t,i), \\
u(0,i) = u_0(i) \text{ and } v(0,i) = v_0(i), \qquad i \in \mathbb{Z}.
\end{cases}
$$

- **Prey:** $u = u(t,i)$ and **Predator:** $v = v(t,i)$.
- **Initial data u_0 and v_0 are both compactly supported.**

Predator-prey systems

Aim: Study the spreading speeds for the prey-predator system for $t > 0$, $i \in \mathbb{Z}$

$$\begin{cases} \frac{d}{dt}u(t,i) = \sum_{j\in\mathbb{Z}} J_1(t,j)[u(t,i-j)-u(t,i)] \\ \qquad +r(t)u(t,i)\big(1-u(t,i)\big)-p(t)u(t,i)v(t,i), \\ \frac{d}{dt}v(t,i) = \sum_{j\in\mathbb{Z}} J_2(t,j)[v(t,i-j)-v(t,i)] \\ \qquad +q(t)u(t,i)v(t,i)-\nu(t)v(t,i), \\ u(0,i)=u_0(i) \text{ and } v(0,i)=v_0(i), \qquad i\in\mathbb{Z}. \end{cases}$$

- **Prey:** $u = u(t,i)$ and **Predator:** $v = v(t,i)$.
- **Initial data u_0 and v_0 are both compactly supported.**

All coefficients are uniformly continuous and uniformly positive; all admit a mean value and $\inf_{t\geq 0}\big(q(t)-\nu(t)\big) > 0$.

Assumptions

As for the Fisher-KPP equation, we assume that the kernels are:
J_k are nonnegative, $J_k \in l^1(\mathbb{Z}, L_t^\infty(0,\infty))$,
thin-tailed and symmetric, $\inf_{t\geq 0} J_k(t,\pm 1) > 0$,
mean value and

$$\lambda^{-1} \sum_{j\in\mathbb{Z}} \langle J_k(\cdot, j)\rangle e^{\lambda j} \to \infty \text{ as } \lambda \to abs(J_k).$$

Two linear speeds

- If $v \equiv 0$, then u satisfies

$$\frac{d}{dt}u(t,i) = \sum_{j\in\mathbb{Z}} J_1(t,j)[u(t,i-j)-u(t,i)] + r(t)u(t,i)\big(1-u(t,i)\big).$$

Spreading speed:

$$c_u^* := \inf_{\lambda\in(0,abs(J_1))} \lambda^{-1}\left(\sum_{j\in\mathbb{Z}}\langle J_1(\cdot,j)\rangle[e^{\lambda j}-1] + \langle r\rangle\right).$$

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Spreading speed:

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- If $u \equiv 1$, then v satisfies

$$\frac{d}{dt}v(t,i) = \sum_{j\in\mathbb{Z}} J_2(t,j)[v(t,i-j)-v(t,i)] + \left(q(t)-\nu(t)\right)v(t,i).$$

Spreading speed:

$$c_v^* := \inf_{\gamma\in(0,abs(J_2))} \gamma^{-1}\left(\sum_{j\in\mathbb{Z}}\langle J_2(\cdot,j)\rangle[e^{\gamma j}-1] + \langle q-\nu\rangle\right).$$

Main results: slow predator case i.e. $c_u^* > c_v^*$

Theorem (Slow predator case)

Assume $c_u^* > c_v^*$. If the initial data are compactly supported then (u, v) satisfies:

(i) $\lim_{t\to\infty} \sup_{|i|\geq ct} u(t,i) = 0, \forall c > c_u^*;$

(ii) for all $c_v^* < c_1 < c_2 < c_u^*$ and for all $c > c_v^*$ one has:

$$\lim_{t\to\infty} \sup_{c_1 t\leq |i|\leq c_2 t} |1 - u(t,i)| = 0 \text{ and } \lim_{t\to\infty} \sup_{|i|\geq ct} v(t,i) = 0;$$

(iii) for all $c \in [0, c_v^*)$ one has:

$$\liminf_{t\to\infty} \inf_{|i|\leq ct} v(t,i) > 0.$$

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$$\lim_{t\to\infty} \sup_{c_1 t\leq |i|\leq c_2 t} |1-u(t,i)| = 0 \text{ and } \lim_{t\to\infty} \sup_{|i|\geq ct} v(t,i) = 0;$$

(iii) for all $c \in [0, c_v^*)$ one has:

$$\liminf_{t\to\infty} \inf_{|i|\leq ct} v(t,i) > 0.$$

Main results: fast predator case i.e. $c_u^* \leq c_v^*$

Theorem (Fast predator case)

Assume $c_u^* \leq c_v^*$ *and* (u_0, v_0) *are compactly supported. Then* (u, v) *satisfies:*

(i) $\lim_{t\to\infty} \sup_{|i|\geq ct} [u(t,i) + v(t,i)] = 0, \forall c > c_u^*;$

(ii) for all $c \in [0, c_u^*)$ *one has:*

$$\liminf_{t\to\infty} \inf_{|i|\leq ct} v(t,i) > 0,$$

$$0 < \liminf_{t\to\infty} \inf_{|i|\leq ct} u(t,i) \leq \limsup_{t\to\infty} \sup_{|i|\leq ct} u(t,i) < 1.$$

Main results: fast predator case i.e. $c_u^* \leq c_v^*$

Theorem (Fast predator case)

Assume $c_u^* \leq c_v^*$ *and* (u_0, v_0) *are compactly supported. Then* (u, v) *satisfies:*

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(ii) for all $c \in [0, c_u^*)$ *one has:*

$$\liminf_{t\to\infty} \inf_{|i|\leq ct} v(t,i) > 0,$$

$$0 < \liminf_{t\to\infty} \inf_{|i|\leq ct} u(t,i) \leq \limsup_{t\to\infty} \sup_{|i|\leq ct} u(t,i) < 1.$$

Sketch of proof: comparision with F-KPP equation

Since $u \approx 0$ implies $v \approx 0$ we obtain

Lemma (Pointwise estimates–(i))

For each $\delta > 0$, there exist $M_\delta > 0$ and $T_\delta > 0$ s.t.

$$v(t,i) \leq \delta + M_\delta u(t,i), \quad \forall t \geq T_\delta, \forall i \in \mathbb{Z}.$$

Sketch of proof: comparision with F-KPP equation

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For each $\delta > 0$, there exist $M_\delta > 0$ and $T_\delta > 0$ s.t.

$$v(t,i) \leq \delta + M_\delta u(t,i), \quad \forall t \geq T_\delta, \forall i \in \mathbb{Z}.$$

$$\frac{d}{dt}u(t,i) = \sum_{j\in\mathbb{Z}} J_1(t,j)[u(t,i-j) - u(t,i)] + r(t)u(1-u) - p(t)uv,$$

$$\geq \sum_{j\in\mathbb{Z}} J_1(t,j)[u(t,i-j) - u(t,i)] + r(t)u(1-u) - p(t)u(\delta + Mu)$$

Hence u spreads faster than speed c_u^*.

Sketch of proof

If $c < c_u^*$ and $v \approx 0$ then $u \approx 1$.

Lemma (Pointwise estimates–(ii))

Fix $c \in [0, c_u^*)$, for each $\alpha > 0$, there exist $M_\alpha > 0$ and $T_\alpha > 0$ s.t.

$$1 - u(t,i) \leq \alpha + M_\alpha v(t,i), \quad \forall t \geq T_\alpha, \forall |i| \leq ct.$$

Sketch of proof

If $c < c_u^*$ and $v \approx 0$ then $u \approx 1$.

Lemma (Pointwise estimates–(ii))

Fix $c \in [0, c_u^*)$*, for each* $\alpha > 0$*, there exist* $M_\alpha > 0$ *and* $T_\alpha > 0$ *s.t.*

$$1 - u(t,i) \leq \alpha + M_\alpha v(t,i), \quad \forall t \geq T_\alpha, \forall |i| \leq ct.$$

$$\frac{d}{dt}v(t,i) = \sum_{j\in\mathbb{Z}} J_2(t,j)[v(t,i-j) - v(t,i)] + q(t)uv - \nu(t)v$$

$$\geq \sum_{j\in\mathbb{Z}} J_2(t,j)[v(t,i-j) - v(t,i)] + v(q(t) - \nu(t) - \alpha - Mv), \ |i| \leq ct.$$

KPP type equation on a growing domain $|i| \leq ct$.

Thank you for your attention.

"Ecocultural range-expansion model of modern humans in Paleolithic"

Joe Yuichiro Wakano (Meiji University, Japan)

Modern human range expansion and the resulting extinction or assimilation of archaic humans such as Neanderthals took place roughly 50,000 years ago. This phenomenon is recently very actively studied by using genetic methods such as ancient DNA analysis. In this talk, a reaction diffusion model is proposed with emphasis on archaeological and ecological aspects. Range expansion dynamics are studied as the traveling wave solutions in the system.

This PDF is modified from the original slides

Ecocultural range-expansion model of modern humans in Paleolithic

Joe Yuichiro Wakano

School of Interdisciplinary Mathematical Sciences, Meiji Univ

in collaboration with

Kenichi Aoki (Meiji Univ), Seiji Kadowaki (Nagoya Univ)

William Gilpin, Marcus Feldman (Stanford Univ)

2023/10/31 - 11/2 ICMMA2023 (10/31 10:40-11:20)

"**International Conference on "Reaction-diffusion systems: from the past to the future“**

— in memory of Prof. Masayasu Mimura —

Self introduction of J. Y. Wakano and my memory of Mayan

- 1998–2001 Ph.D student (Centre for Ecological Research, Kyoto Univ)
  - Evolutionary game theory, models in behavioral ecology, etc.
  - Japan Association for Mathematical Biology (JAMB), found in 1989
    1989–1992 Chief: N. Shigesada, 1992–1994 Chief: **M. Mimura**

- 2001–2007 post-Doc (Univ. of Tokyo)
  - Pattern formation of bacterial colony
  - 2001 JAMB-SMB Joint Conference on Mathematical Biology（Hawaii, Hiro）
  - 2003 JSMB found (Japanese Society for Mathematical Biology)
    Frequent visit to Mimura-sensei's lab at Ikuta Campus, Meiji University

- 2004 Mayan became a professor at School of Science and Technology, Meiji Univ.
  - His ambition: foundation of a new research area

- 2007–2013 Specially appointed associate professor, Meiji Univ.
  - Meiji Institute for Advanced Study of Mathematical Sciences (MIMS), found in 2007
  - Daishin Ueyama (professor) and H. Izuhara (grad student)

- 2008–2012
  - Global COE Program "Formation and Development of Mathematical Sciences Based on Modeling and Analysis"（現象数理学の形成と発展） Leader: **M. Mimura**
  - Mayan, Daishin, and I were central members to prepare the application
  - Mayan looked so nervous before and after the interview. He looked so happy when we got a good news
  - K. Ikeda（post-Doc）

Self introduction of J. Y. Wakano and my memory of Mayan

- 2009-2012
  - PRESTO (さきがけ) "Innovative model of biological processes and its development"
    Mayan was one of advisory professors, and pushed me hard to win this grant
  - Face-to-face lessons on how to talk, a treasure in my life

- 2011 Graduate School of Advanced Mathematical Sciences, Meiji Univ
  - Y. Tanaka (a grad student)

- 2013 Interdisciplinary Mathematical Sciences, Meiji Univ
  - Department of Mathematical Sciences Based on Modeling and Analysis（現象数理学科）

- 2013-2014 JSMB President: **M. Mimura** Chief: J .Y. Wakano

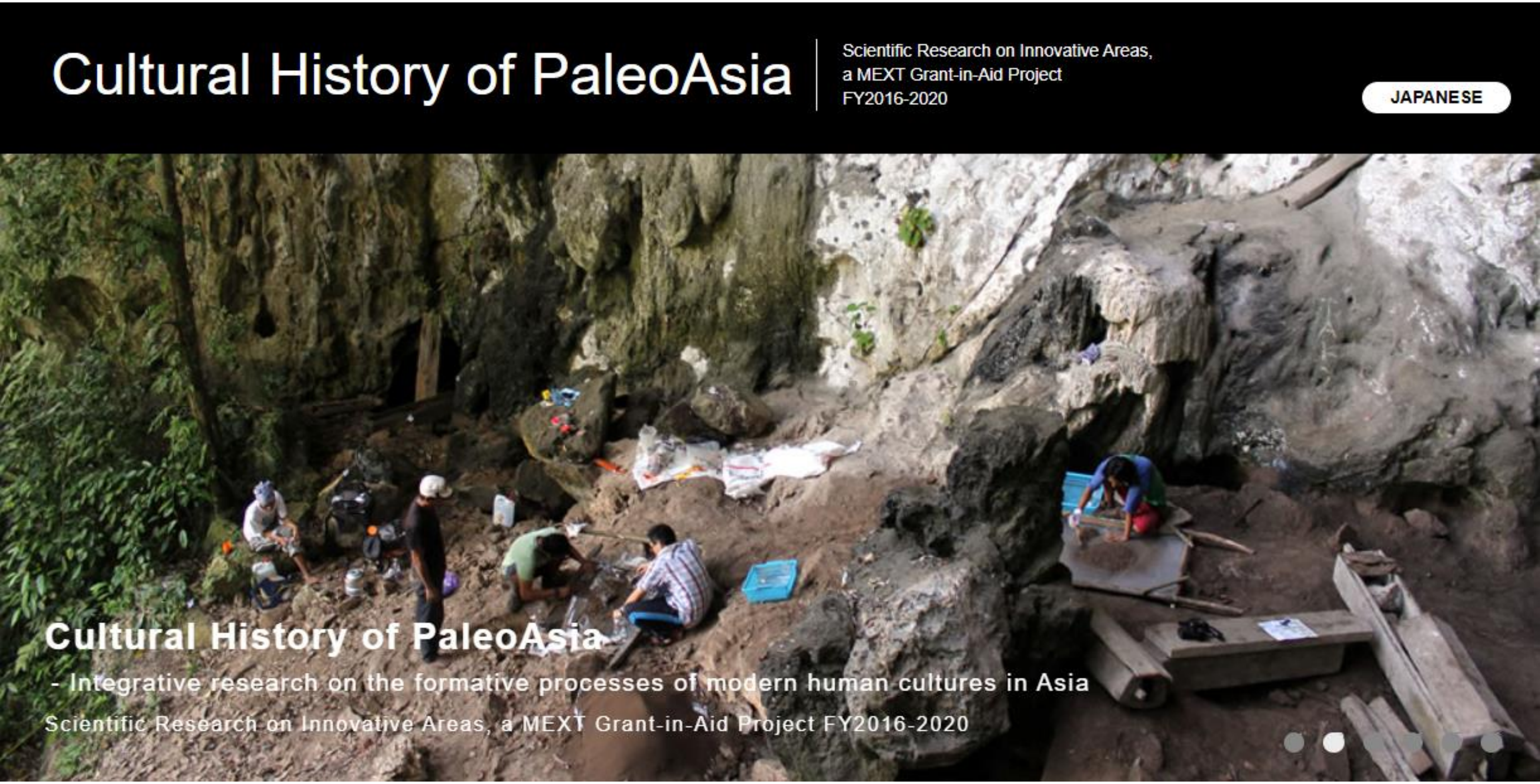
- 2021 Mayan passed away

Zu J, Mimura M & Wakano JY (2010) The evolution of phenotypic traits in a predator-preysystem subject to Allee effect. *Journal of Theoretical Biology* 262:528-543

Wakano JY, Ikeda K, Miki T & Mimura M (2011) Effective dispersal rate is a function of habitat size and corridor shape: mechanistic formulation of a two-patch compartment model for spatially continuous systems. *Oikos* 120: 1712-1720

Scotti T, Mimura M & Wakano JY (2015) Avoiding Toxic Prey May Promote Harmful Algal Blooms. *Ecological Complexity* 21:157-165

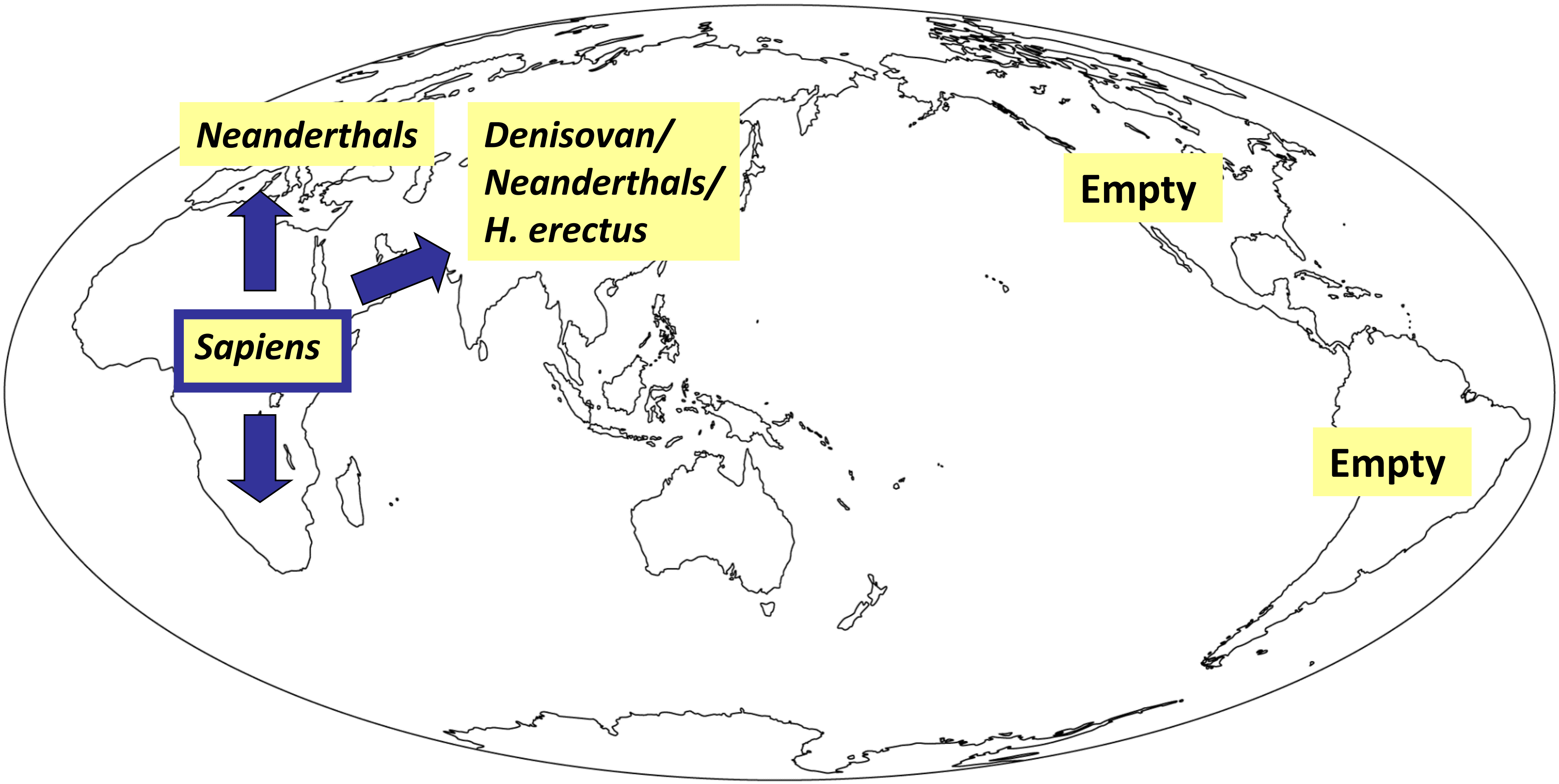
joint project of archeology, environmental sciences, cultural anthropology, and **<u>mathematical modeling and analysis</u>**



Integrative research on the formative processes of modern human cultures in Asia

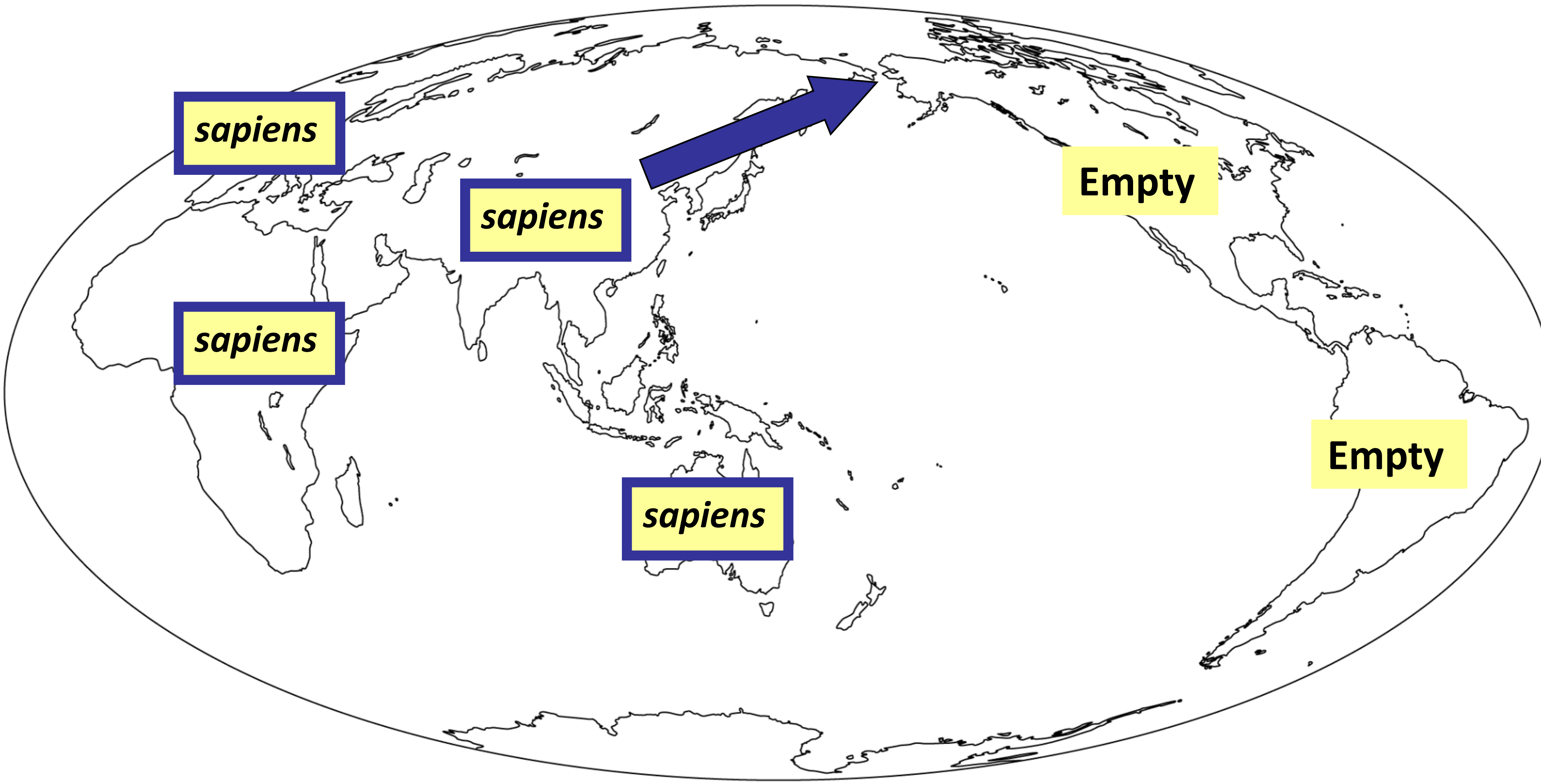
Range expansion of modern humans

Second wave of out-of-Africa 100～50 Kya.



Range expansion of modern humans

Extinction of archaic humans ～30 Kya.



Range expansion of modern humans

Oppenheimer 2012

Pruffer et al. (2014)

LETTER

Slon et al. (2018)

https://doi.org/10.1038/s41586-018-0455-x

The genome of the offspring of a Neanderthal mother and a Denisovan father

Viviane Slon[1,7]*, Fabrizio Mafessoni[1,7], Benjamin Vernot[1,7], Cesare de Filippo[1], Steffi Grote[1], Bence Viola[2,3], Mateja Hajdinjak[1], Stéphane Peyrégne[1], Sarah Nagel[1], Samantha Brown[4], Katerina Douka[4,5], Tom Higham[5], Maxim B. Kozlikin[3], Michael V. Shunkov[3,6], Anatoly P. Derevianko[3], Janet Kelso[1], Matthias Meyer[1], Kay Prüfer[1] & Svante Pääbo[1]*

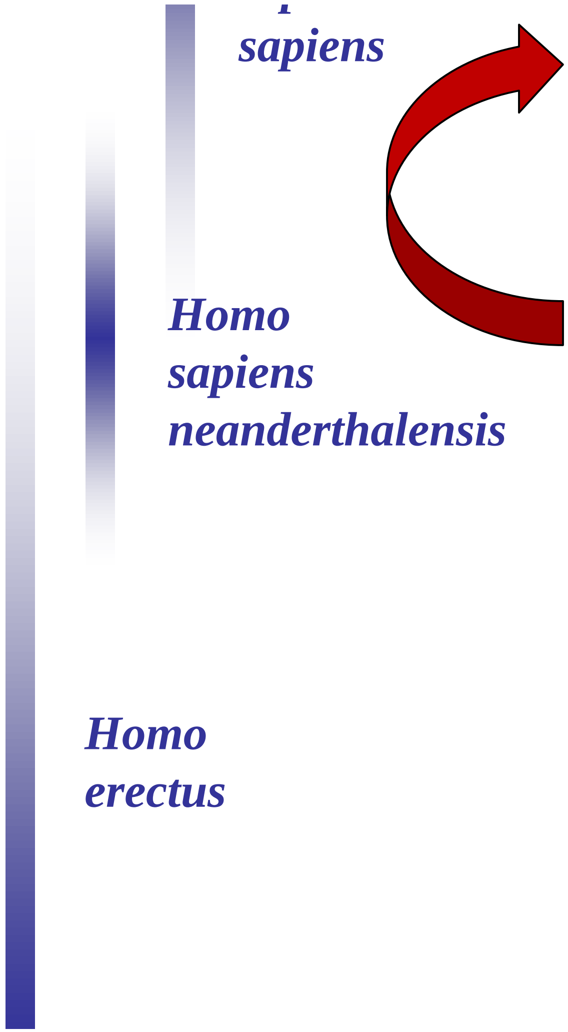
Svante Paebo (2022 Nobel Prize)

Bae et al. (2017, Science)

Cultural Evolution in Paleolithic

10kya

Upper Paleolithic transition（上部旧石器革命）



50kya

250kya

Spatial dynamics of genes (species)
vs
Spatial dynamics of culture

- Culture is not completely determined by genes (species)
  - cultural inheritance is not identical to genetic inheritance

- "Modern humans" by their DNAs
- "Modern humans" by their behavior and culture

- Here we define modern humans by their DNAs, and aim to study how spatial dynamics of modern human culture are related to those of modern humans

What makes cultural differences ? learning hypothesis

- Innate (genetic) difference in learning ability
  - Wakano & Miura (2014), Aoki & Feldman (2014), Lehmann et al. (2013), Aoki et al. (2012), … and much more

modern humans

Neanderthals

smart ??

stupid ??

What makes cultural differences ?
population size hypothesis

- Innate (genetic) difference in learning ability
  - Wakano & Miura (2014), Aoki & Feldman (2014), Lehmann et al. (2013), Aoki et al. (2012), … and much more

- Small population results in low culture
  - Henrich (2004) claims that the loss of adaptive culture is triggered by decreased population size by using empirical data in Tasmania and analyzing mathematical models.
  - population decrease ⇒ lower culture
  - population increase ⇒ higher culture
  - Many theoretical studies (Shennan 2001; Henrich 2004; Strimling et al. 2009; Mesoudi 2011; Lehmann et al. 2011; Aoki et al. 2011; Kobayashi & Aoki 2012; Forgaty et al 2015)

Positive feedback loop between culture and population size

- Low/high culture results in small/large population
  - Previous studies
    - Ghirlanda & Enquist 2007; Aoki 2015; Gilpin et al. 2016
  - Positive spiral
    - population increase ⇒ higher culture ⇒ population even increases ⇒ ...
  - Negative spiral
    - population decrease ⇒ lower culture ⇒ population even decreases ⇒ ...

- Different states can appear in genetically homogeneous population.

- Previous models of positive feedback loop did not explicitly model spatial structure

Single species model

Model Assumptions

- Culture of an individual is either **skilled** or **non-skilled**
  - Newborns socially learn culture from a random individual of the same species at the same location
  - Skill is lost at rate γ (bad memory, lack of practice)
  - Non-skilled changes to skilled at rate δ (individual learning)

- **Carrying capacity** is increased when local population carries more skilled individuals
  - Skill gives benefit to local population
  - Population size hypothesis

- Individuals randomly migrate in 1-D continuous space
  - modeled by diffusion equation
  - migration of skilled individuals results in spatial spread of skills

Reaction diffusion system (single species case)

population density (skilled + non-skilled)

$$\frac{\partial}{\partial t}N(x,t) = D\frac{\partial^2}{\partial x^2}N + rN\left[1 - \frac{N}{M(Z)}\right]$$

density of skilled individuals

$$\frac{\partial}{\partial t}Z(x,t) = D\frac{\partial^2}{\partial x^2}Z + rZ\left[1 - \frac{N}{M(Z)}\right] - \gamma Z + \delta(N - Z)$$

skilled changes to non-skilled (term $-\gamma Z$)

increase of skilled due to newborns learning skill (term $rZ\left[1 - \frac{N}{M(Z)}\right]$)

non-skilled changes to skilled (term $\delta(N - Z)$)

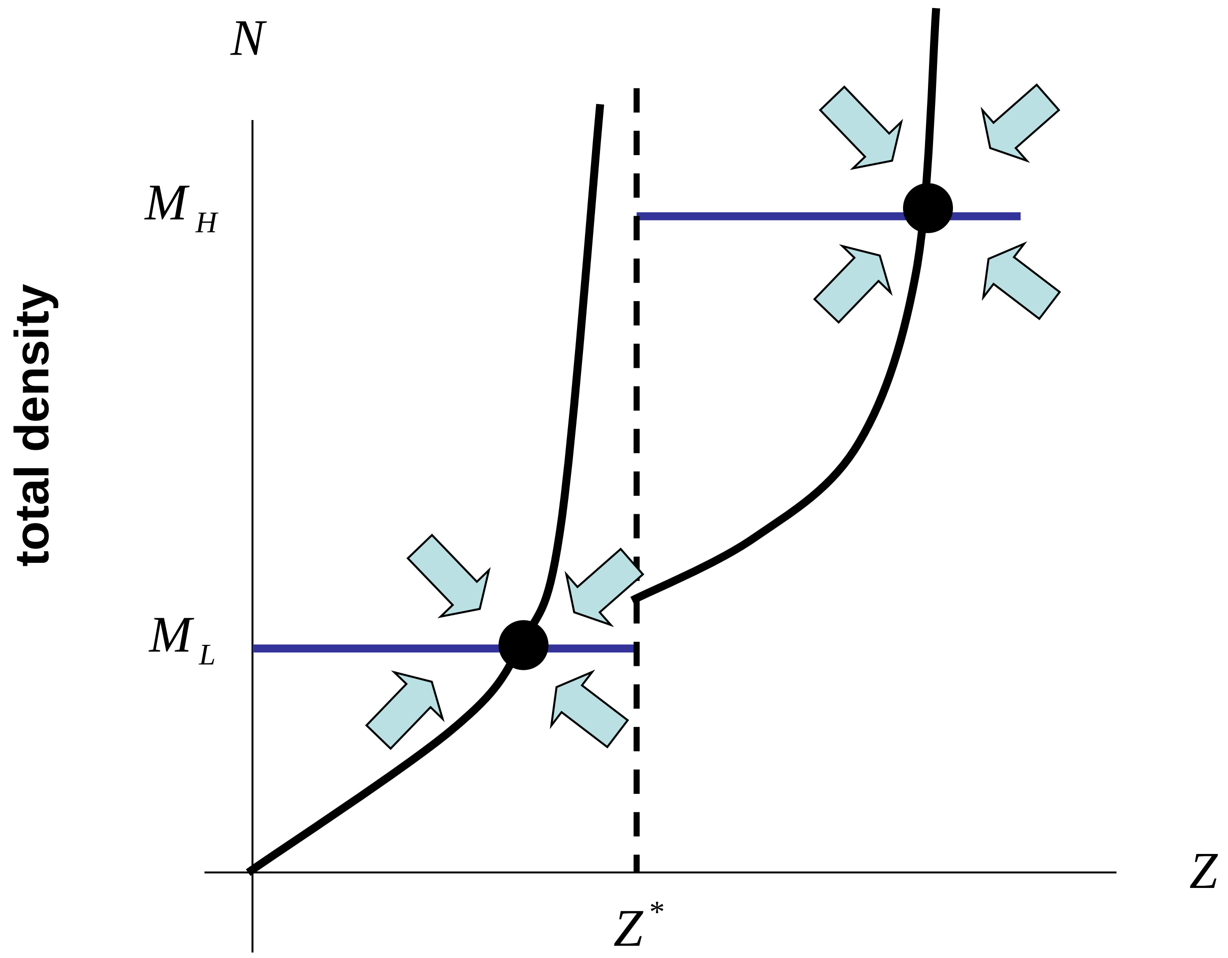
carrying capacity

$$M(Z) = \begin{cases} M_L & (Z < Z^*) \\ M_H & (Z \geq Z^*) \end{cases}$$

Low, when skilled < Z*

High, when skilled > Z*

Critical density of skilled, Z*



density of skilled individuals

two locally stable equilibria

$$(N, Z) = (M_L, \theta M_L)$$
$$(N, Z) = (M_H, \theta M_H)$$

$$\theta = \frac{\delta}{\gamma + \delta}$$

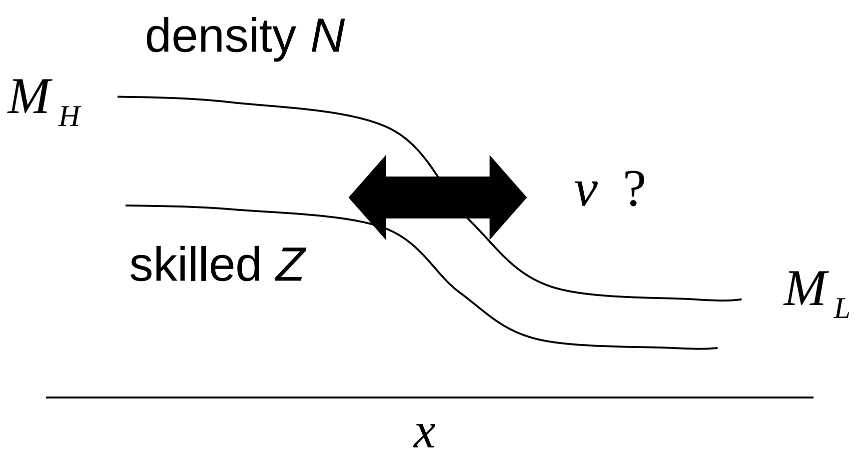
Traveling wave solution (TWS)

competition between populations at "high" and "low" equilibriums

$$\frac{\partial}{\partial t}N(x,t) = D\frac{\partial^2}{\partial x^2}N + rN\left[1 - \frac{N}{M(Z)}\right]$$

$$\frac{\partial}{\partial t}Z(x,t) = D\frac{\partial^2}{\partial x^2}Z + rZ\left[1 - \frac{N}{M(Z)}\right] - \gamma Z + \delta(N - Z)$$

「High density +
High skilled density」



「Low density +
Low skilled density」

Special solution (invariant manifold)

$$\frac{\partial}{\partial t} N(x,t) = D\frac{\partial^2}{\partial x^2} N + rN\left[1 - \frac{N}{M(Z)}\right]$$

$$\frac{\partial}{\partial t} Z(x,t) = D\frac{\partial^2}{\partial x^2} Z + rZ\left[1 - \frac{N}{M(Z)}\right] - \gamma Z + \delta(N - Z)$$

$$Z(x,t) = \theta N(x,t)$$

$$N(x,t) = u(x,t)$$

$$Z(x,t) = \theta u(x,t)$$

is a special solution of the system

$$\theta = \frac{\delta}{\gamma + \delta}$$

innovation-forgetting balance

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru\left[1 - \frac{u}{M(\theta u)}\right]$$

This invariant manifold is globally attracting in PDE sense (maximum principle).

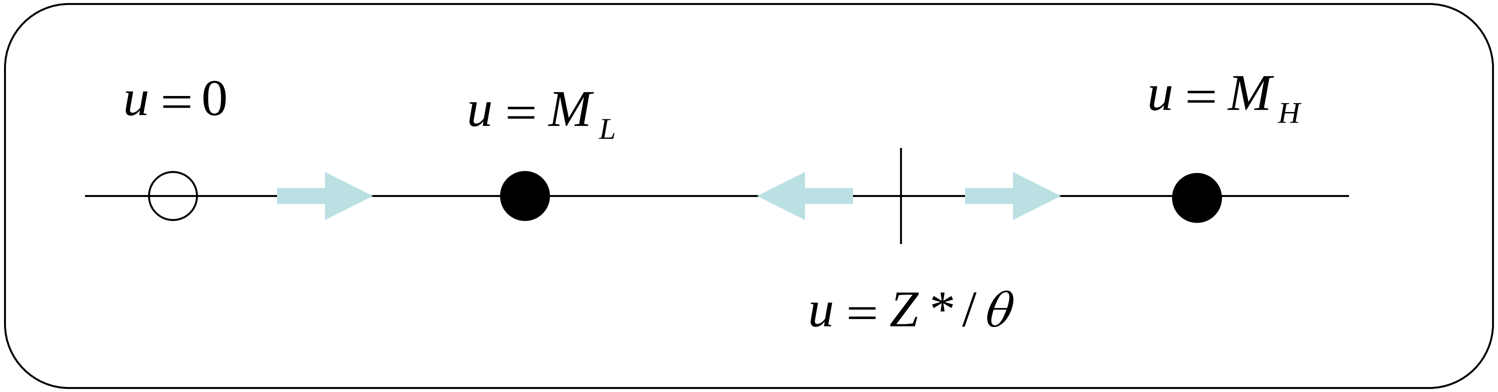
Bistable TWS

e.g.) Allen-Cahn eq.

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru\left[1 - \frac{u}{M(\theta u)}\right]$$

$$M(Z) = \begin{cases} M_L & (Z < Z^*) \\ M_H & (Z \ge Z^*) \end{cases}$$

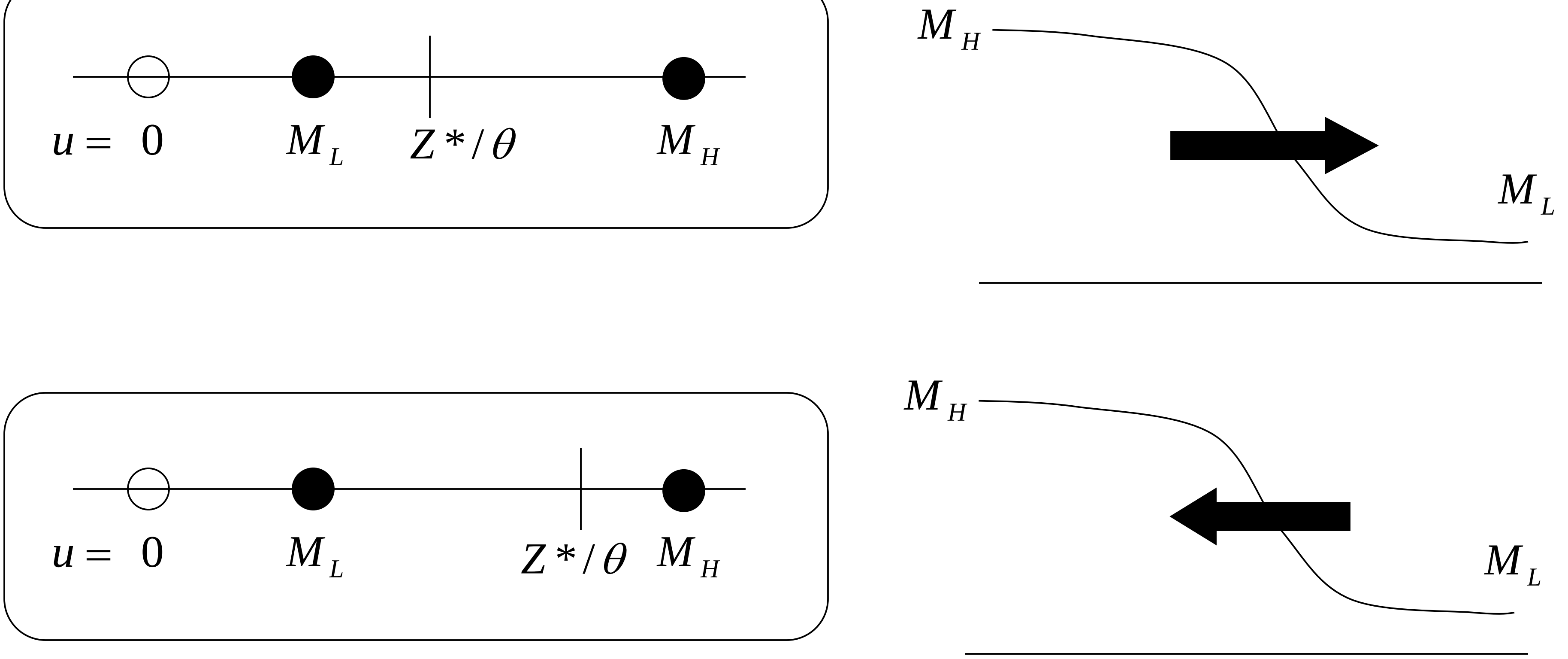
Reaction term dynamics



The direction of TWS is determined by the sign of

$$\int_{M_L}^{z^*/\theta} u\left(1 - \frac{u}{M_L}\right)du + \int_{z^*/\theta}^{M_H} u\left(1 - \frac{u}{M_H}\right)du = \left(\frac{M_H{}^2 - M_L{}^2}{6}\right) - \frac{1}{3}\left(\frac{1}{M_L} - \frac{1}{M_H}\right)\left(\frac{Z^*}{\theta}\right)^3$$

Traveling wave can progress in **either** direction



The high equilibrium does not always win.

Both high and low equilibria are **locally** stable.

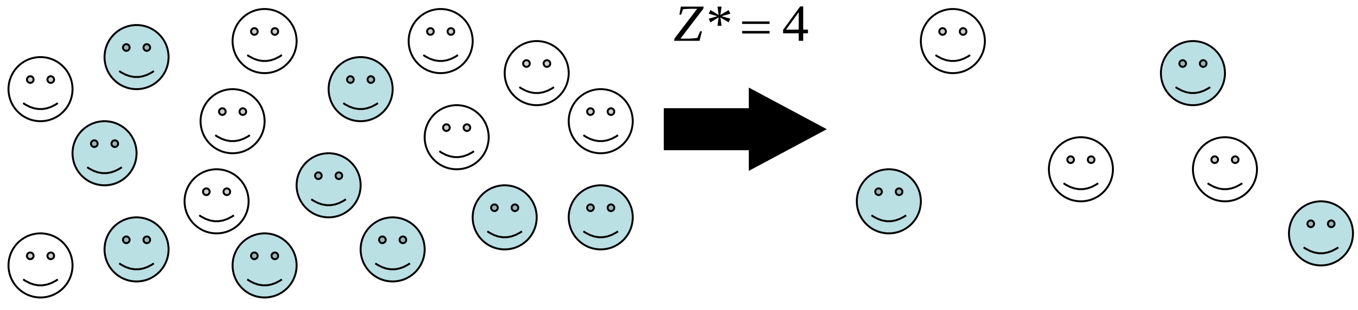
$M_H = 18$

9 skilled + 9 non-skilled

$M_L = 6$

3 skilled + 3 non-skilled

$Z^* = 4$



Africa *Asia*

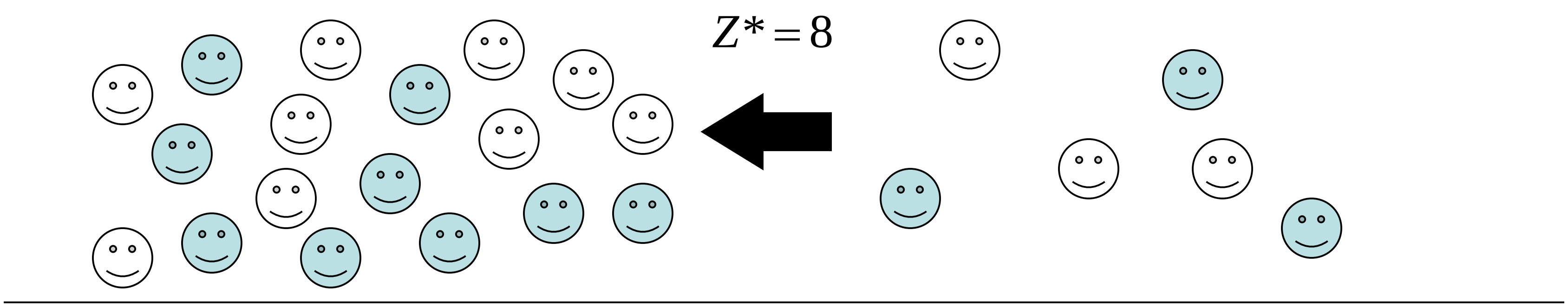
When they start to randomly migrate,
high equilibrium (advanced culture) does or does not spread.

$M_H = 18$

9 skilled + 9 non-skilled

$M_L = 6$

3 skilled + 3 non-skilled



Africa *Asia*

When they start to randomly migrate,
high equilibrium (advanced culture) does or does not spread.

Condition for high equilibrium to spatially invade low equilibrium

$$Z^* < \theta \left[\frac{M_L M_H (M_H + M_L)}{2} \right]^{\frac{1}{3}}$$

$$\downarrow \qquad \theta = \frac{\delta}{\gamma + \delta}, \quad M_H = \alpha M_L = \alpha K$$

$$\frac{Z^*}{K} < \left(\frac{\delta}{\gamma + \delta} \right) \left[\frac{\alpha (1 + \alpha)}{2} \right]^{\frac{1}{3}}$$

- When lower number of skilled is required
  to transition to high carrying capacity
- or when skill is easier to obtain / harder to lose
- or when skill has larger impact on carrying capacity

Two species model

(additional) Assumptions

- Two species: modern humans and Neanderthals

- No innate difference in abilities of the two species
  - the same learning ability, the same demographic ability

- Within each species, carrying capacity is increased when local population carries more skilled individuals

- Ecological resource competition between the two species
  - weaker than intraspecific competition
  - different species use difference niches with some overlap

Lotka-Volterra type reaction diffusion system (Neanderthal vs. modern humans)

Neanderthals

$$\frac{\partial}{\partial t} N_1(x,t) = D \frac{\partial^2}{\partial x^2} N_1 + rN_1 \left[1 - \frac{N_1 + bN_2}{M(Z_1)} \right]$$

$$\frac{\partial}{\partial t} Z_1(x,t) = D \frac{\partial^2}{\partial x^2} Z_1 + rZ_1 \left[1 - \frac{N_1 + bN_2}{M(Z_1)} \right] - \gamma Z_1 + \delta (N_1 - Z_1)$$

modern humans

$$\frac{\partial}{\partial t} N_2(x,t) = D \frac{\partial^2}{\partial x^2} N_2 + rN_2 \left[1 - \frac{N_2 + bN_1}{M(Z_2)} \right]$$

$$\frac{\partial}{\partial t} Z_2(x,t) = D \frac{\partial^2}{\partial x^2} Z_2 + rZ_2 \left[1 - \frac{N_2 + bN_1}{M(Z_2)} \right] - \gamma Z_2 + \delta (N_2 - Z_2)$$

(Intraspecific) Carrying capacity

$$M(Z) = \begin{cases} M_L & (Z < Z^*) \\ M_H & (Z \geq Z^*) \end{cases}$$

(Interspecific) Competition coefficient $0 < b < 1$

niche overlap

$$\frac{\partial}{\partial t}N_1(x,t)=D\frac{\partial^2}{\partial x^2}N_1+rN_1\left[1-\frac{N_1+bN_2}{M(Z_1)}\right]$$

$$\frac{\partial}{\partial t}Z_1(x,t)=D\frac{\partial^2}{\partial x^2}Z_1+rZ_1\left[1-\frac{N_1+bN_2}{M(Z_1)}\right]-\gamma Z_1+\delta(N_1-Z_1)$$

$$\frac{\partial}{\partial t}N_2(x,t)=D\frac{\partial^2}{\partial x^2}N_2+rN_2\left[1-\frac{N_2+bN_1}{M(Z_2)}\right]$$

$$\frac{\partial}{\partial t}Z_2(x,t)=D\frac{\partial^2}{\partial x^2}Z_2+rZ_2\left[1-\frac{N_2+bN_1}{M(Z_2)}\right]-\gamma Z_2+\delta(N_2-Z_2)$$

$$Z_i(x,t)=\theta N_i(x,t)$$

$$A_i(x,t):=\frac{Z_i(x,t)}{N_i(x,t)}$$

$$\frac{\partial}{\partial t}A_i(x,t)=\delta(1-\frac{1}{\theta}A_i)+D\frac{N_i\frac{\partial Z_i^2}{\partial x^2}-Z_i\frac{\partial N_i}{\partial x^2}}{N_i^2}$$

Global attractor exists: $\lim_{t\to\infty}A_i(x,t)=\theta$

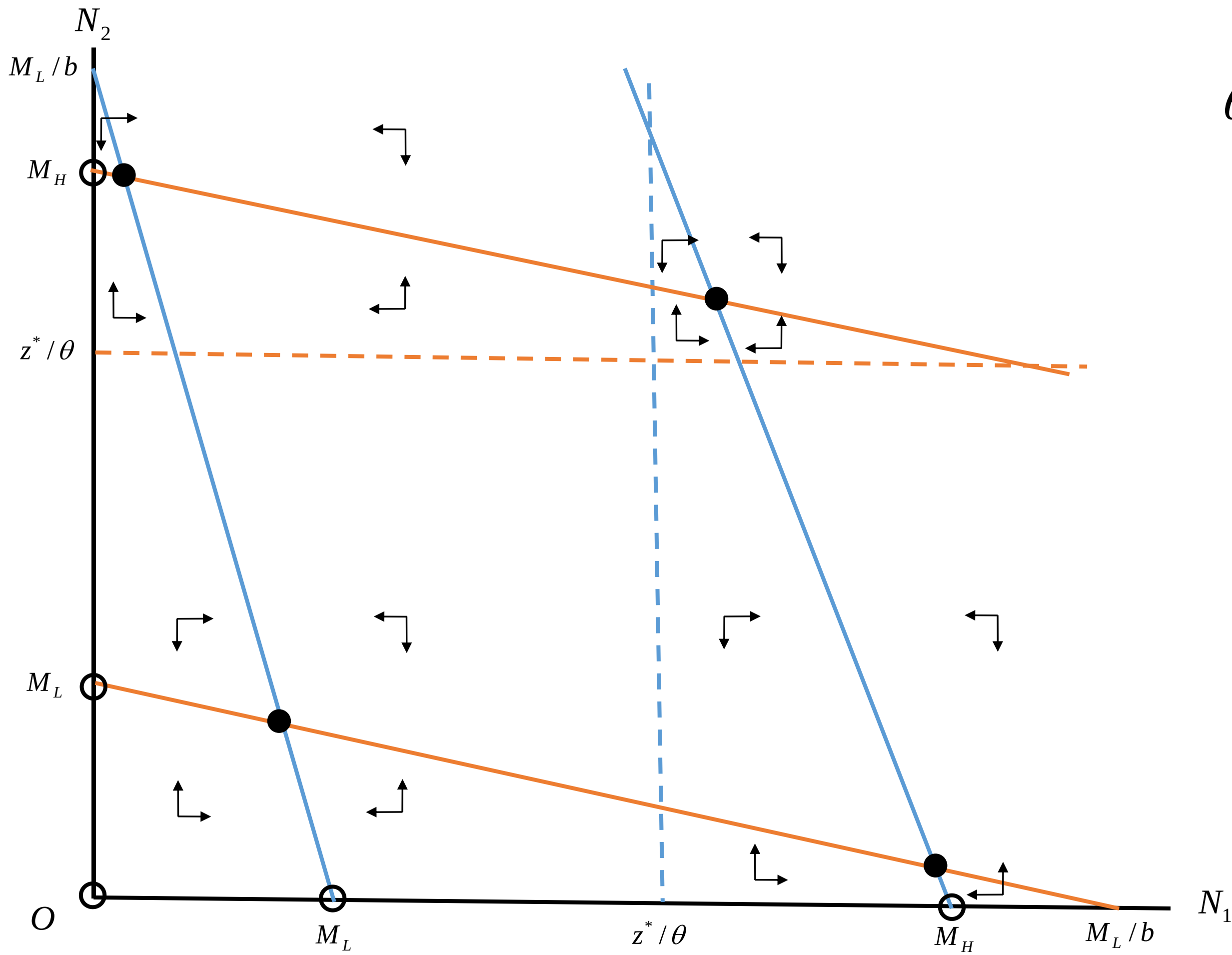
innovation-forgetting balance

$$\frac{\partial N_1}{\partial t}=D\frac{\partial^2 N_1}{\partial x^2}+rN_1\left[1-\frac{N_1+bN_2}{M(\theta N_1)}\right]$$

$$\frac{\partial N_2}{\partial t}=D\frac{\partial^2 N_2}{\partial x^2}+rN_2\left[1-\frac{N_2+bN_1}{M(\theta N_2)}\right]$$

When interspecific competition (b) is very weak

$$M_L < \frac{z^*}{\theta} < M_H < \frac{M_L}{b}$$

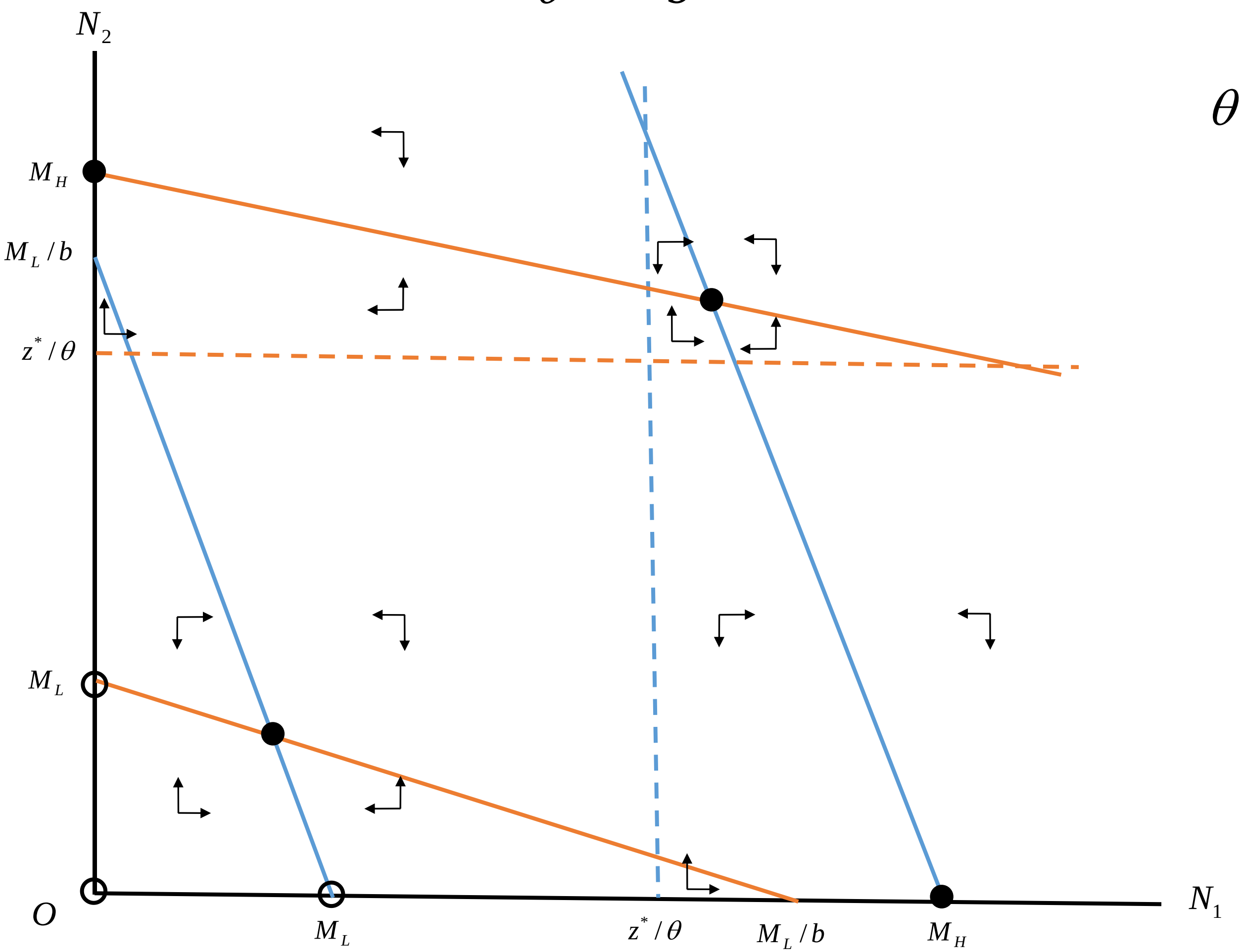


$$\theta = \frac{\delta}{\delta + \gamma}$$

When interspecific competition (*b*) is intermediate

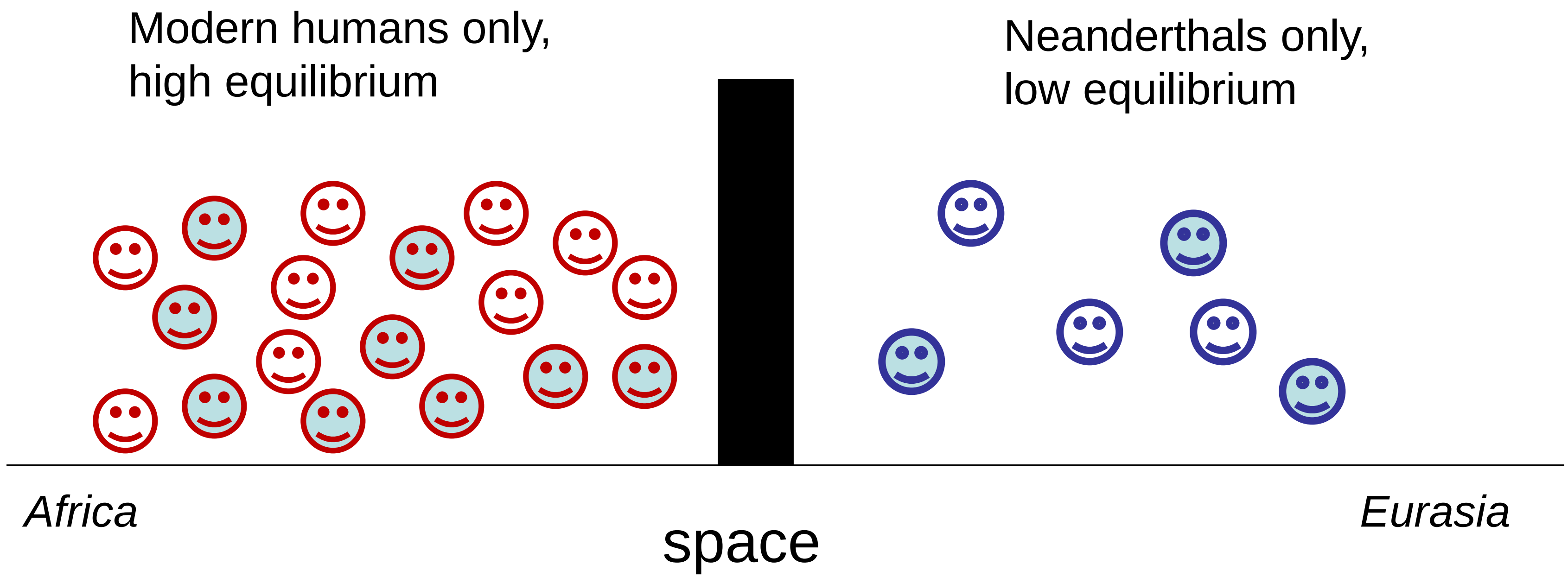
$$M_L < \frac{z^*}{\theta} < \frac{M_L}{b} < M_H$$

$$\theta = \frac{\delta}{\delta + \gamma}$$



Spatial dynamics of population and culture

- Depending on initial distribution, various dynamics occur.
- For explanation, we initially set



(For simplicity, we only consider cases when high equilibrium defeats low.)

Initial condition (t=0)

Skilled densities (Z_1, Z_2) are not shown since we always observed quick convergence to

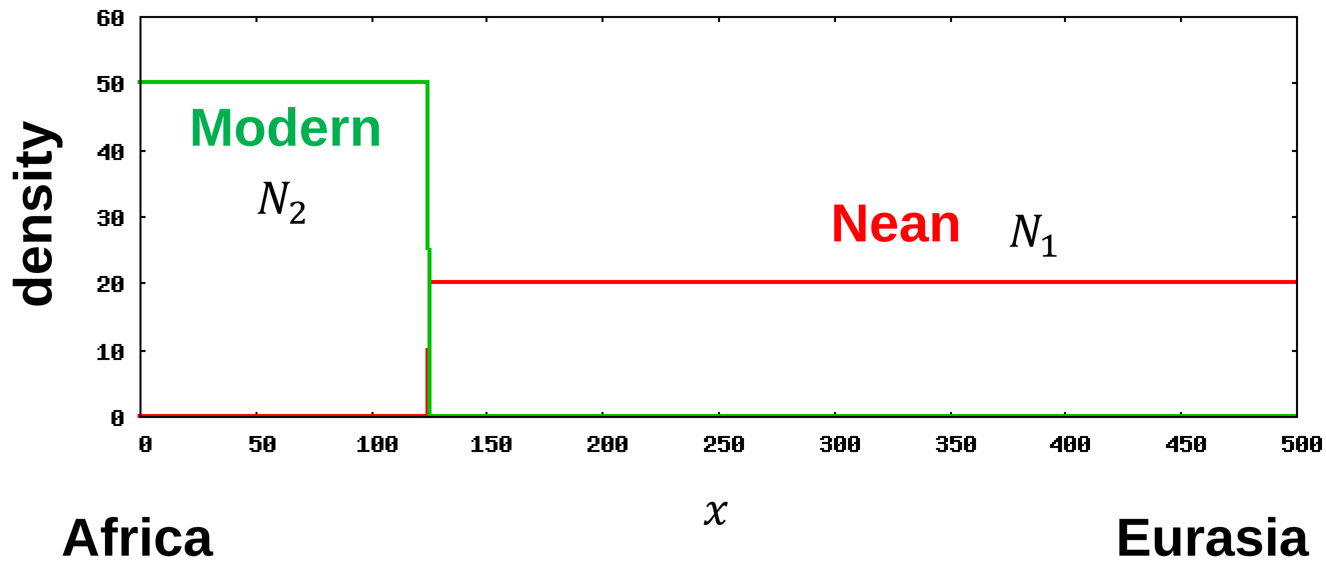
$$Z_i(x,t) = \theta N_i(x,t)$$

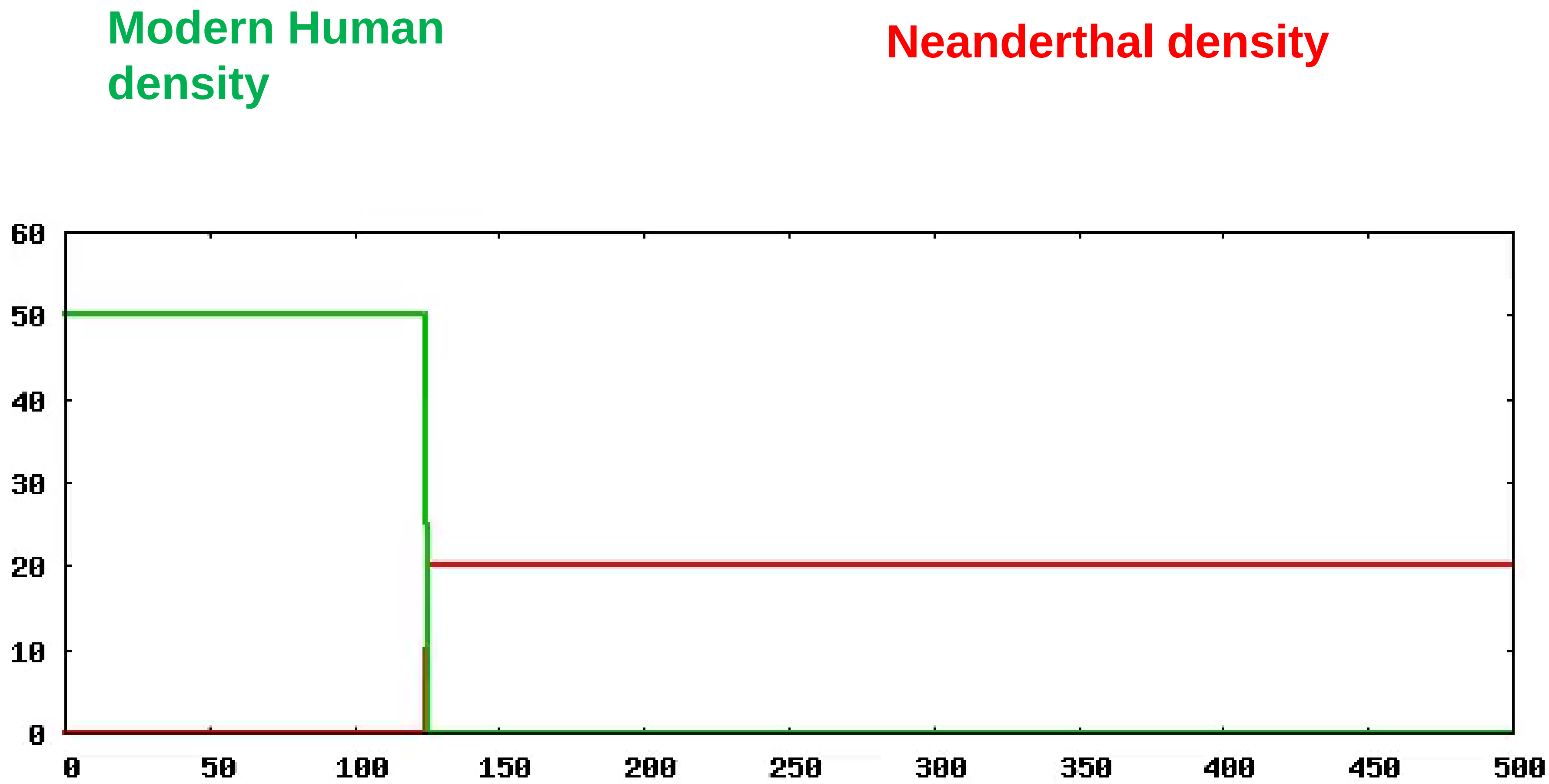
African side (left)

No Neanderthals. Modern humans at high-skill-high-density equilibrium

Eurasian side (right)

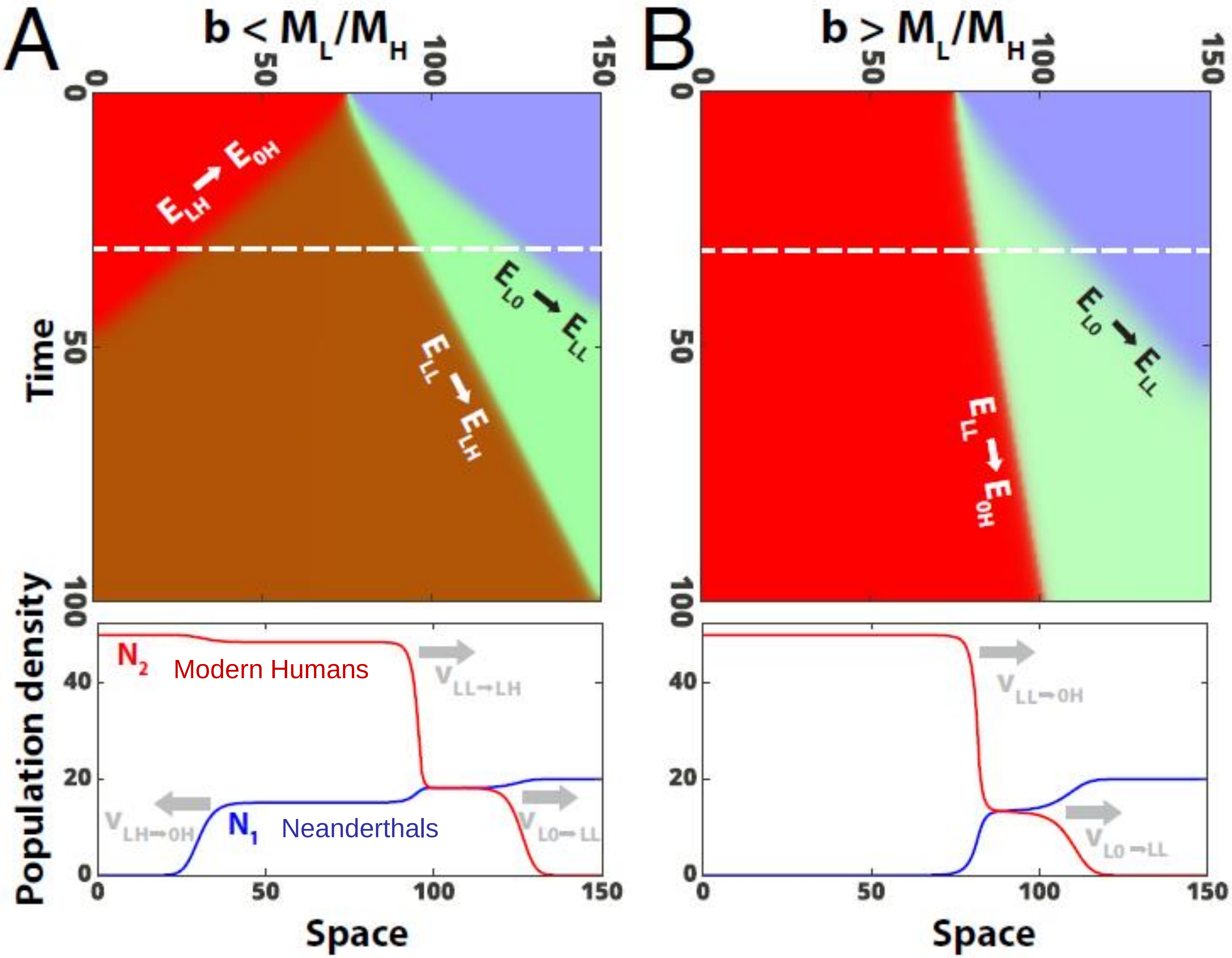
Neanderthals at low-skill-low-density equilibrium. No modern humans.

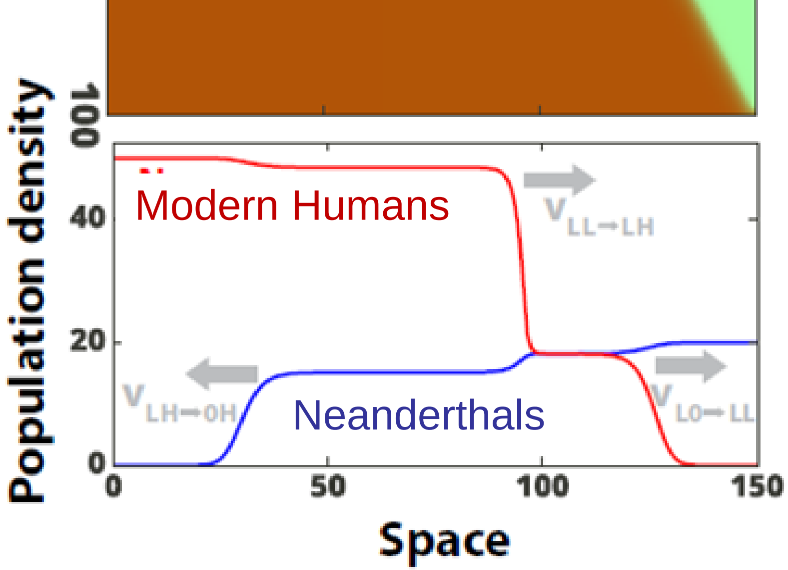



Modern Human density
Neanderthal density
60
50
40
30
20
10
0
0
50
100
150
200
250
300
350
400
450
500

interspecific competition, *b*

very weak intermediate





Case A : very small *b*

Between the initial two states,
two additional states appear

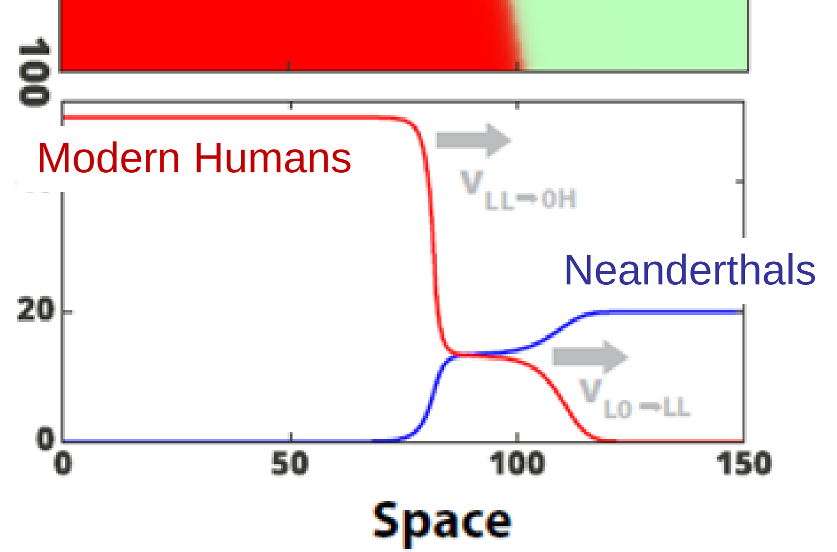
(N, MH) = (low, high) : Final state, coexistence of the two species

(N, MH) = (low, low) : Temporal state
This state can last very long at locations very far away from the initial contact

Temporal symmetric coexistence might accelerate genetic interbreeding.
Final asymmetric coexistence might result in complete assimilation of Neanderthals by modern humans.

Very weak interspecific competition ⇒ assimilation scenario

Case B : intermediate *b*



Between the initial two states,
one additional state appear

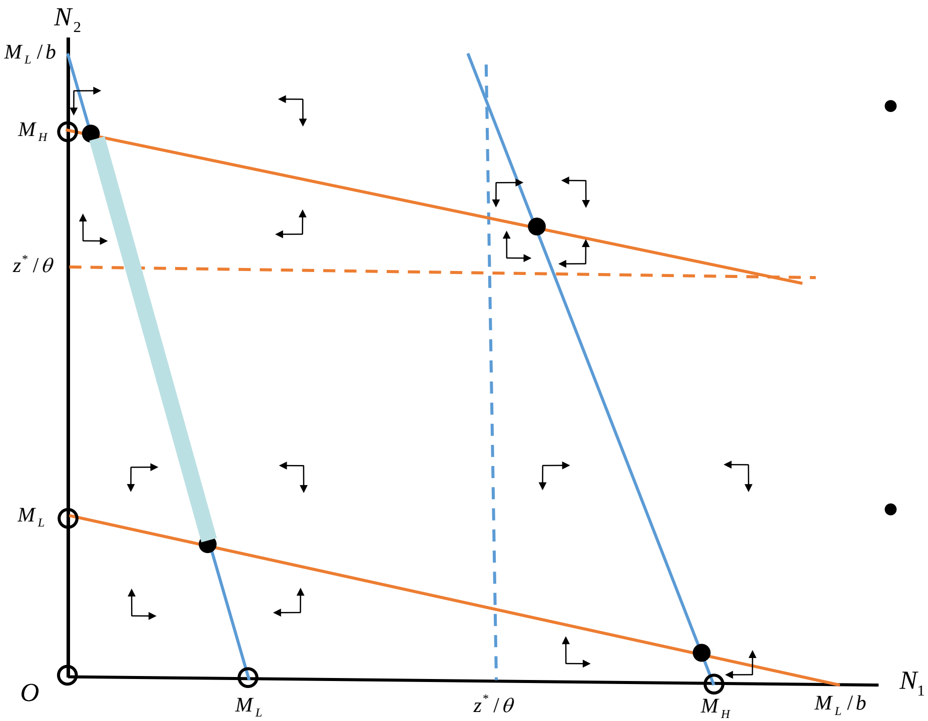
(N, MH) = (zero, high) : Final state

(N, MH) = (low, low) : Temporal state
This state can last very long at locations very far away from the initial contact

Finally, modern humans replace Neanderthals.
Local temporal coexistence persists long only at locations very far.
Limited genetic interbreeding is expected.

Intermediate interspecific competition ⇒ Replacement

The direction of the second wave: bistable case



- Approximating a TWS trajectory by a line segment, we obtain

$$\frac{\partial N_2}{\partial t} = D\frac{\partial^2 N_2}{\partial x^2} + rN_2\left[1-\frac{N_2+b(M_L-bN_2)}{M(\theta N_2)}\right]$$

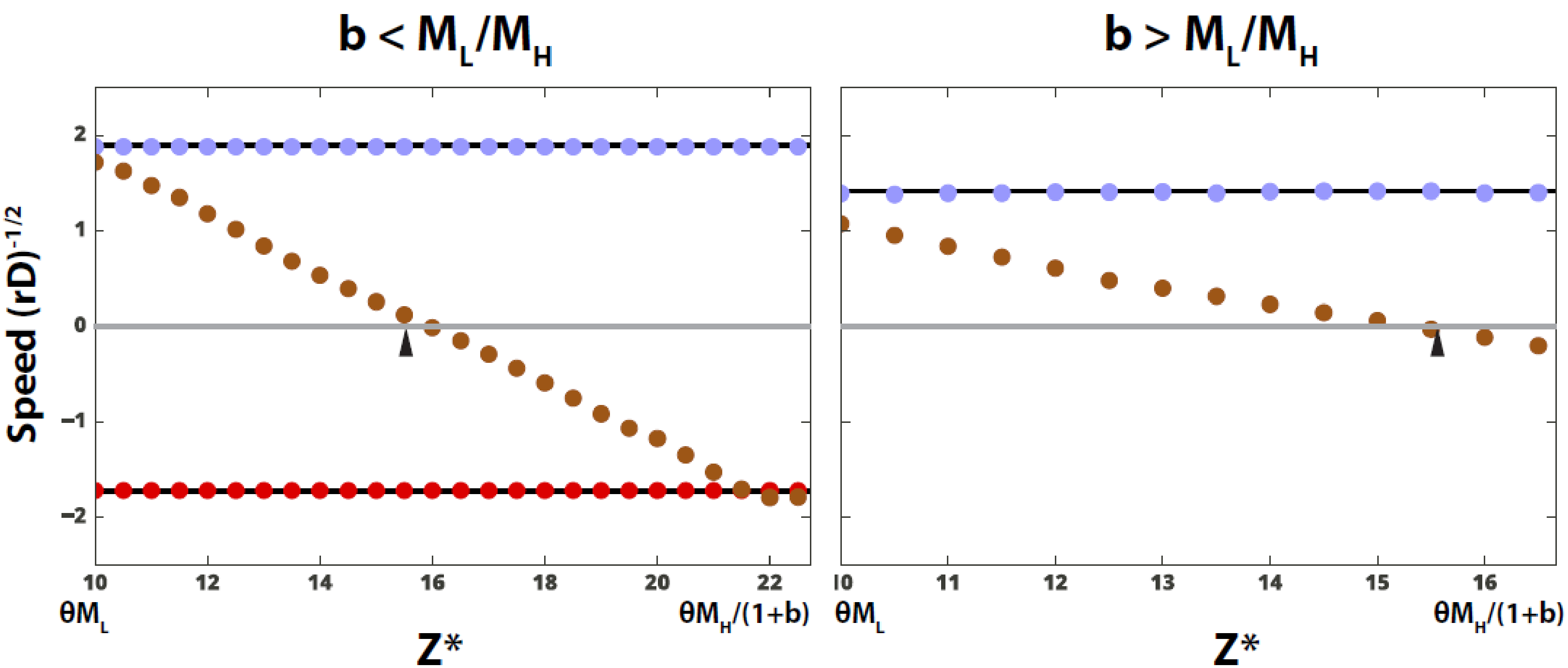
- This yields the (approximate) condition for 'high' equilibrium spatially spreads:

$$P(z^*) = \int_{M_L/(1+b)}^{(M_H-bM_L)/(1-b^2)} rn_2\left[1-\frac{n_2+b(M_L-bn_2)}{M_2(\theta_2 n_2)}\right]dn_2$$

High density region spreads in TWS if $\frac{(1-b^2)Z^*}{\theta}$ is small s.t.

$$2\left(\frac{(1-b^2)Z^*}{\theta}\right)^3 + 3bM_L\left(\frac{(1-b^2)Z^*}{\theta}\right)^2 - M_L\left(M_H{}^2 + (1-3b)M_H M_L + b^3 M_L{}^2\right) < 0$$

Numerics agree with these analytic predictions



linear conjecture speed

analytically approximated Z* value
for which zero-speed wave exits

Comparison with archaeological data

Bladelets

Early Epipaleolithic
Tor Hamar F (n = 4907)

EUP (Early Ahmarian)
Tor Hamar G (n = 4291)

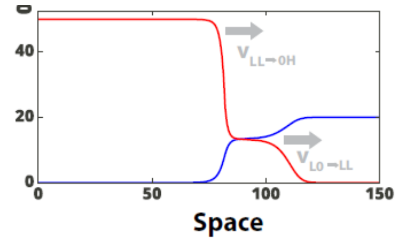
IUP-EUP
Tor Fawaz (n = 1494)

IUP (Emiran)
Wadi Aghar (Kadowaki 2017)
(n = 317)

Late MP
Tor Faraj
(n = 767)

new

old

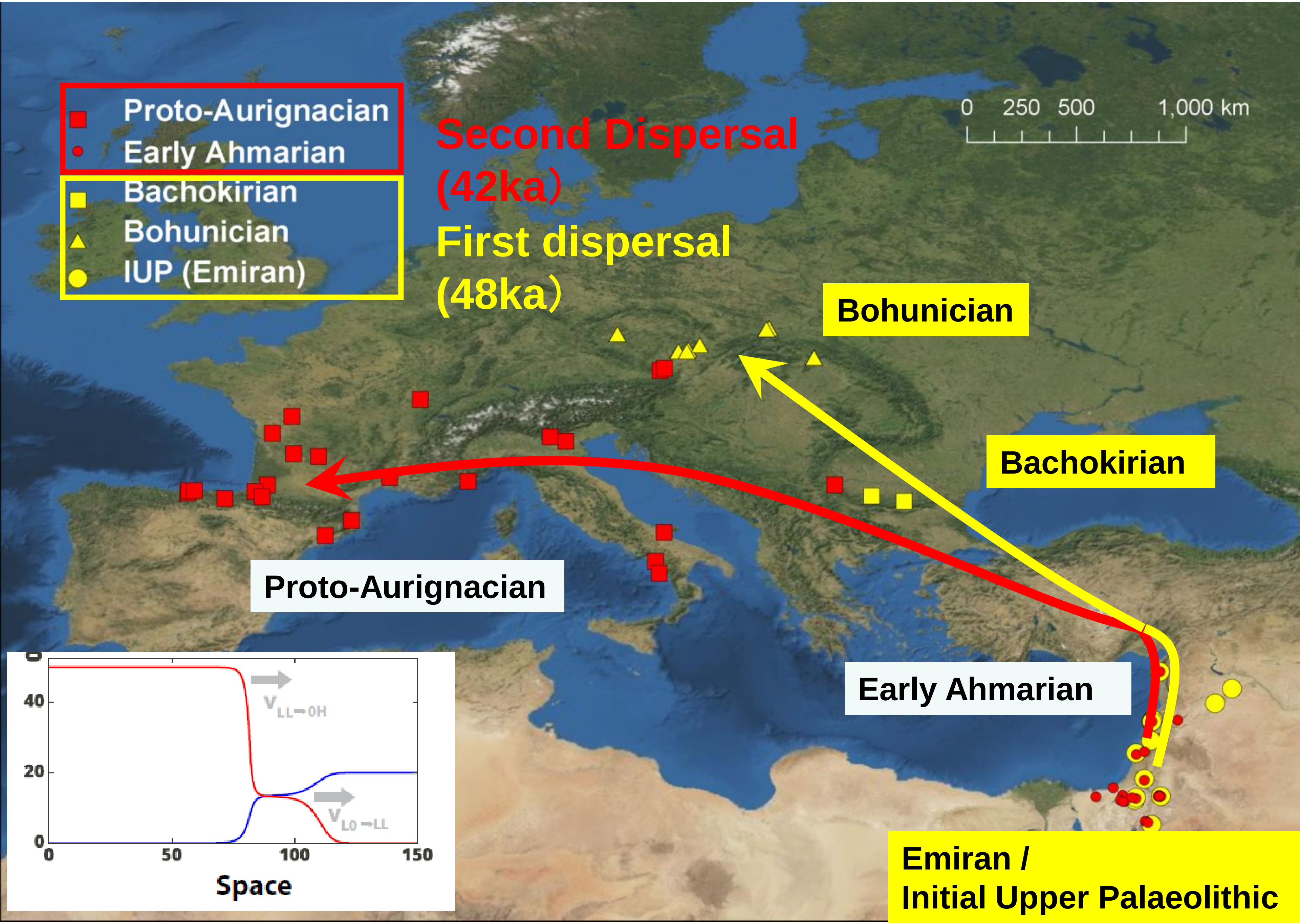


Two dispersals of modern humans

Archaeological records from Levant to Europe

Hubrin 2015

kya



Proto-Aurignacian
Early Ahmarian
Bachokirian
Bohunician
IUP (Emiran)
Second Dispersal
(42ka)
First dispersal
(48ka)
0 250 500 1,000 km
Bohunician
Bachokirian
Proto-Aurignacian
Early Ahmarian
Emiran /
Initial Upper Palaeolithic
40
20
0
0
50
100
150
Space

First Dispersal (Initial Upper Paleolithic)

Emiran（Levant）

Bohunician（Europe)

Belfer-Cohen and Goring-Morris 2007

Skrdla 2003 (Stranska skala)

Second Dispersal (42ka)

Mellars 2011 (*Nature, 479*)

Protoaurignacian
(Europe)

Early Ahmarian
(Levant)

Bladelet

Bladelet

Bordes 2006
(Le Piage level K)

Goring-Morris and Belfer Cohen 2006

Summary

- A model of spatial dynamics of human population and culture
  - Positive feedback loop between population density and culture
  - Cognitive and demographic **equivalence** of Neanderthals and modern humans
  - They interact through ecological resource competition
- High-density-high-culture equilibrium does not always spatially spread even if it is locally stable
- Raplacement of Neanderthals by modern humans is possible when ecological niche overlap is high ($M_L/M_H<b<1$)

- Range-expansion of modern human might have occurred in **two major waves** with different speeds
  - "First" wave : Ecological invasion due to different niche utilization
  - "Second" wave : Driven by high-density-high-culture equilibrium

Thank you

Wakano JY, Gilpin W, Kadowaki S, Feldman MW, Aoki K (2018) Ecocultural range-expansion scenarios for the replacement or assimilation of Neanderthals by modern humans
Theoretical Population Biology 119:3-14

"Traveling wave solutions for a three-species diffusive competition system"

Jong-Shenq Guo (Tamkang University, Taiwan)

We shall discuss the existence and stability of traveling waves for a three-species diffusive competition system. This talk is based on some recent joint works with Karen Guo and Masahiko Shimojo.

Traveling wave solutions for a three-species diffusive competition system

Jong-Shenq Guo (TKU)

Tamsui, New Taipei City, Taiwan

Joint work with K. Guo (PU) and M. Shimojo (TMU)

§1. Introduction

Consider the following diffusive 3-species competition system

$$\begin{cases} u_t = u_{xx} + r_1 u(1 - u - a_2 v - a_3 w), \ x \in \mathbb{R}, \ t > 0, \\ v_t = v_{xx} + r_2 v(1 - b_1 u - v - b_3 w), \ x \in \mathbb{R}, \ t > 0, \\ w_t = w_{xx} + r_3 w(1 - c_1 u - c_2 v - w), \ x \in \mathbb{R}, \ t > 0, \end{cases} \tag{1}$$

where u, v, w are the densities of three competing species.

- r_1, r_2, r_3 the intrinsic growth rates, and $a_2, a_3,\ b_1, b_3,\ c_1, c_2$ the interspecific competition coefficients are positive
- the carrying capacity of each species is normalized to be 1
- we assume the diffusivities of all species are equal to 1

- Due to the normalization of the carrying capacities, that $a_2 > 1 (a_2 < 1)$ means v is a strong (weak) competitor to u.
- For example, $\{a_2 > 1, b_1 > 1\}$ means u, v are strong competing species, etc.
- When $b_3 < 1$ and $c_2 < 1$, we have the semi-coexistence state $E_c^u := (0, v_c, w_c)$:

$$v_c = \frac{1 - b_3}{1 - b_3 c_2} \in (0,1), \;\; w_c = \frac{1 - c_2}{1 - b_3 c_2} \in (0,1).$$

- It is also possible to have the co-existence state $E_* := (u_*, v_*, w_*)$ with $u_*, v_*, w_* \in (0,1)$.

Traveling wave solution

A traveling wave of (1) is a solution (u, v, w) of (1) in the form

$$(u, v, w)(x, t) = (\phi_1, \phi_2, \phi_3)(z),\ z := x - st,$$

for some constant (the wave speed) $s \in \mathbb{R}$ and some functions (the wave profiles) $\{\phi_1, \phi_2, \phi_3\}$.

Problem: find unknown $\{s, \phi_1, \phi_2, \phi_3\}$ such that

$$\begin{cases} \phi_1'' + s\phi_1' + r_1\phi_1(1 - \phi_1 - a_2\phi_2 - a_3\phi_3) = 0,\ z \in \mathbb{R}, \\ \phi_2'' + s\phi_2' + r_2\phi_2(1 - b_1\phi_1 - \phi_2 - b_3\phi_3) = 0,\ z \in \mathbb{R}, \\ \phi_3'' + s\phi_3' + r_3\phi_3(1 - c_1\phi_1 - c_2\phi_2 - \phi_3) = 0,\ z \in \mathbb{R}. \end{cases} \quad (2)$$

- Two asymptotic states:

$$(\phi_1, \phi_2, \phi_3)(\pm\infty) = E_\pm, \tag{3}$$

  where $E_\pm$ are two different constant states of (1).
- The wave is called a monostable wave if one of $\{E_\pm\}$ is unstable and the other is stable
- It is bistable if both states $E_\pm$ are stable
- When $s > 0$, we call E_- the invading state and E_+ the invaded state. Roles are exchanged if $s < 0$.
- Difficulty: comparison principle does not hold for 3-species competition systems.

Mimura's works on 3-species competition systems

- Chen, Hung, Mimura, Tohma, Ueyama, Exact travelling wave solutions of three-species competition-diffusion systems, DCDS-B (2012)
- Chen, Hung, Mimura, Tohma, Ueyama, Semi-exact equilibrium solutions for three-species competition-diffusion systems, Hiroshima Math. J. (2013)
- Mimura, Tohma, Dynamic coexistence in a three-species competition-diffusion system, Ecol. Compl. (2015)
- Contento, Mimura, Tohma, Two dimensional travelling waves arising from planar front interaction in a three species competition diffusion system, JJIAM (2015)
- Contento, Hilhorst, Mimura, Ecological invasion in competition-diffusion systems when the exotic species is either very strong or very weak, J. Math. Biol. (2018)
- Chang, Chen, Hung, Mimura, Ogawa, Existence and stability of non-monotone travelling wave solutions for the diffusive Lotka-Volterra system of three competing species, Nonlinearity (2020)

In the work in 2020 (Nonlinearity) for general d_i, they study

- u, w are strong competitors: $a_3 > 1, b_1 > 1, b_3 > 1, c_1 > 1$
- v is a very weak competitor: $0 < a_2, c_2 \ll 1$
- TWS with $E_- = (1, 0, 0),\ E_+ = (0, 0, 1)$: a bistable wave

by the bifurcation theory and the method of super-sub-solutions

- See also Chang, Chen, JDDE (2023) for a related work.
- The sign of wave speed is an open question?

Our aim: monostable waves?? easier!

§2-1. Main results: existence

- Assume there are two aboriginal weak competing species v and w in the sense $b_3 < 1$ and $c_2 < 1$.
- Let u be an alien species.
- Assume that the semi-coexistence state $E_c^u = (0, v_c, w_c)$ is unstable for the ODE system of (1):

$$\beta := 1 - a_2 v_c - a_3 w_c > 0. \tag{4}$$

- Note that condition (4) can be achieved, e.g., when

$$a_2 + a_3 < 1. \tag{5}$$

Theorem 1

Let $b_3 < 1$ *and* $c_2 < 1$. *Suppose, in addition to condition* (4), *that*

$$r_1\beta \geq \max\{r_2(b_1 + b_3c_2v_c), r_3[c_1 + c_2(1 - v_c)]\}. \tag{6}$$

Set $s_* := 2\sqrt{r_1\beta}$. *Then there is a positive solution* (ϕ_1, ϕ_2, ϕ_3) *of* (2) *satisfying*

$$(\phi_1, \phi_2, \phi_3)(+\infty) = E_c^u = (0, v_c, w_c)$$

for any $s \geq s_*$.

- Assume w is an aboriginal weak competitor.
- Let $\sigma_+ := \max\{0, \sigma\}$ for $\sigma \in \mathbb{R}$.

Theorem 2

Let $a_3 < 1$ *and* $b_3 < 1$. *Assume*

$$r_2(1 - b_3) = r_1(1 - a_3) \geq r_3(c_1 + c_2 - 1)_+. \tag{7}$$

Set $s^* := 2\sqrt{r_1(1 - a_3)}$. *Then there exists a positive solution* (ϕ_1, ϕ_2, ϕ_3) *of* (2) *satisfying*

$$(\phi_1, \phi_2, \phi_3)(+\infty) = E_w = (0, 0, 1)$$

for any $s \geq s^*$.

A strong alien competitor u is introduced to the habitat of two aboriginal competing species v and w:

Theorem 3

Let $\{s, (\phi_1, \phi_2, \phi_3)\}$ *be a traveling wave obtained in Theorem 1 with* $s \geq s_*$ *or Theorem 2 with* $s \geq s^*$. *Then* $(\phi_1, \phi_2, \phi_3)(-\infty) = E_u = (1, 0, 0)$, *if we assume*

$$a_2 b_1 \geq 1,\ a_3 c_1 \geq 1,\ a_2 + a_3 < 1. \tag{8}$$

- Theorem 3 shows that system (1) has traveling waves of mixed front-pulse type connecting E_w and E_u for any $s \geq s^*$, under conditions in Theorem 2 and (8).

- Three species are all weak competitors:

Under the assumption

$$a_2 + a_3 < 1,\ b_1 + b_3 < 1,\ c_1 + c_2 < 1, \tag{9}$$

the co-existence state $E_* = (u_*, v_*, w_*)$ exists and E_* is stable.

Theorem 4

Let $\{s, (\phi_1, \phi_2, \phi_3)\}$ be a traveling wave obtained in either Theorem 1 with $s \geq s_*$ or Theorem 2 with $s \geq s^*$. Then $(\phi_1, \phi_2, \phi_3)(-\infty) = (u_*, v_*, w_*)$, if condition (9) is enforced.

§2-2. Proofs for existence

- The main idea of the proofs of Theorems 1 and 2 is to construct suitable (generalized) upper-lower solutions to capture the unstable tails E_c^u and E_w at $+\infty$, resp.
- For the derivation of the stable tail limit at $z = -\infty$, we apply the classical method of contracting rectangles (cf. e.g., Huang-Lin (JMAA14), Lin-Ruan (JDDE14), Chen-G.-Yao (JMAA17), G.-Nakamura-Ogiwara-Wu (NA-RWA20), Chen-Giletti-G. (JDE21)).
- Existence can allow non-equal diffusivities.

Definition 5

A pair of continuous functions $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ and $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ is called a pair of upper-lower solutions of (2), if $\overline{\phi}_i''(z)$, $\underline{\phi}_i''(z)$, $\overline{\phi}_i'(z)$, $\underline{\phi}_i'(z)$, $i = 1, 2, 3$, exist such that

$$
\begin{aligned}
\mathcal{U}_1(z) &:= \overline{\phi}_1''(z) + s\overline{\phi}_1'(z) + \overline{\phi}_1(z) g_1(\overline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)(z) \le 0,\\
\mathcal{U}_2(z) &:= \overline{\phi}_2''(z) + s\overline{\phi}_2'(z) + \overline{\phi}_2(z) g_2(\underline{\phi}_1, \overline{\phi}_2, \underline{\phi}_3)(z) \le 0,\\
\mathcal{U}_3(z) &:= \overline{\phi}_3''(z) + s\overline{\phi}_3'(z) + \overline{\phi}_3(z) g_3(\underline{\phi}_1, \underline{\phi}_2, \overline{\phi}_3)(z) \le 0,\\
\mathcal{L}_1(z) &:= \underline{\phi}_1''(z) + s\underline{\phi}_1'(z) + \underline{\phi}_1(z) g_1(\underline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)(z) \ge 0,\\
\mathcal{L}_2(z) &:= \underline{\phi}_2''(z) + s\underline{\phi}_2'(z) + \underline{\phi}_2(z) g_2(\overline{\phi}_1, \underline{\phi}_2, \overline{\phi}_3)(z) \ge 0,\\
\mathcal{L}_3(z) &:= \underline{\phi}_3''(z) + s\underline{\phi}_3'(z) + \underline{\phi}_3(z) g_3(\overline{\phi}_1, \overline{\phi}_2, \underline{\phi}_3)(z) \ge 0
\end{aligned}
$$

hold for $z \in \mathbb{R}$ except for a finite subset E of $\mathbb{R}$.

Proposition 1

Given $s > 0$. *Suppose that* (2) *has a pair of upper-lower solutions* $(\overline{\phi}_1, \overline{\phi}_2, \overline{\phi}_3)$ *and* $(\underline{\phi}_1, \underline{\phi}_2, \underline{\phi}_3)$ *such that* $\underline{\phi}_i \leq \overline{\phi}_i$ *and*

$$\lim_{z \to z^+} \overline{\phi}_i'(z) \leq \lim_{z \to z^-} \overline{\phi}_i'(z), \quad \lim_{z \to z^-} \underline{\phi}_i'(z) \leq \lim_{z \to z^+} \underline{\phi}_i'(z), \ \forall\, z \in E,$$

for $i = 1, 2, 3$. *Then* (2) *has a solution* (ϕ_1, ϕ_2, ϕ_3) *such that* $\underline{\phi}_i \leq \phi_i \leq \overline{\phi}_i$, $i = 1, 2, 3$.

- A pair of upper-lower solutions serves as the upper and lower bounds for the domain of the integral operator corresponding to differential system (2) so that Schauder's fixed point theorem can be applied to obtain a solution of (2) in this domain (an idea of Ma (JDE01)).

The construction of upper-lower solutions relies on the linearization of the equation of alien species in (2) at the unstable state, e.g., $E_+ = E_c^u$:

$$\lambda_j^2 + s\lambda_j + r_1\beta = 0, j = 1, 2,\ \lambda_2 \le \lambda_1 < 0 \ \leftrightarrow \ e^{\lambda_1 z}, ze^{\lambda_1 z}$$

so that for $s > s_*$:

$$\begin{aligned}
\overline{\phi}_1(z) &:= \min\{1, e^{\lambda_1 z}\},\ \underline{\phi}_1(z) := \max\{0, e^{\lambda_1 z} - pe^{\mu z}\},\\
\overline{\phi}_2(z) &:= \min\{1, v_c + (1 - v_c)e^{\lambda_1 z}\},\ \underline{\phi}_2(z) := \max\{0, v_c(1 - e^{\lambda_1 z})\},\\
\overline{\phi}_3(z) &:= \min\{1, w_c + c_2 v_c e^{\lambda_1 z}\},\ \underline{\phi}_3(z) := \max\{0, w_c(1 - e^{\lambda_1 z})\},
\end{aligned}$$

where p and μ are suitably chosen constants.

To derive the stable tail limit at $z = -\infty$, we first let

$$\phi_i^- := \liminf_{z \to -\infty} \phi_i(z),\ \ \phi_i^+ := \limsup_{z \to -\infty} \phi_i(z),\ \ i = 1, 2, 3.$$

Since $\phi_i \geq 0$, $i = 1, 2, 3$, by the maximum principle we have $0 \leq \phi_i \leq 1$, $i = 1, 2, 3$. Hence

$$0 \leq \phi_i^- \leq \phi_i^+ \leq 1,\ \ i = 1, 2, 3.$$

Then we have

$$\phi_1^- \geq \gamma_1 := 1 - a_2 - a_3 > 0, \tag{10}$$

provided $a_2 + a_3 < 1$.

We only give a proof of Theorem 3.

For this, we define $m_1(\theta) := (1-\theta)\gamma_1/2 + \theta,\ \theta \in [0,1]$.
Since $\gamma_1 < 1$, $m_1(\theta)$ is increasing in $\theta \in [0,1]$ such that $m_1(1) = 1$. Let

$$\mathcal{A} := \{\theta \in [0,1) \mid \phi_1^- > m_1(\theta)\}.$$

By (10), $0 \in \mathcal{A}$ and so the quantity $\theta_0 := \sup \mathcal{A}$ is well-defined such that $\theta_0 \in (0,1]$. Then $\phi_1^- \geq m_1(\theta_0)$ and

$$\phi_2^+ \leq \max\{0, 1 - b_1 m_1(\theta_0)\},\ \phi_3^+ \leq \max\{0, 1 - c_1 m_1(\theta_0)\}.$$

Finally, a contradiction argument leads to $\theta_0 = 1$. □

§3-1. Main results: stability

Given positive constants $\{\sigma_i\}$, define a *distance* function (to Φ)

$$\mathcal{K}[U] := \sum_{i=1}^{3} \sigma_i \mathcal{K}_i[U_i], \quad \mathcal{K}_i[U_i] := U_i - \phi_i - \phi_i \ln \frac{U_i}{\phi_i}, \tag{11}$$

for any positive function $U = (U_1, U_2, U_3)$ defined on $\mathbb{R}$.

Note that $\mathcal{K}[U](z) \geq 0$ for all $z \in \mathbb{R}$ and $\mathcal{K}[U](z) = 0$ if and only if $U(z) = \Phi(z)$ for some $z \in \mathbb{R}$.

For a positive constant R, we let

$$\lambda = \lambda(s; R) := \frac{-s + \sqrt{s^2 - 4R}}{2}, \; s \geq 2\sqrt{R}. \tag{12}$$

Suppose that there exists a set of positive constants $\{\sigma_1, \sigma_2, \sigma_3\}$ such that

$$-I := \sum_{i=1}^{3} \sigma_i (u_i - v_i) \{g_i(u) - g_i(v)\} \leq 0 \qquad (13)$$

for all $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathcal{I} := [0, 1]^3$, where

$$\begin{cases} g_1(u_1, u_2, u_3) := r_1(1 - u_1 - a_2 u_2 - a_3 u_3), \\ g_2(u_1, u_2, u_3) := r_2(1 - b_1 u_1 - u_2 - b_3 u_3), \\ g_3(u_1, u_2, u_3) := r_3(1 - c_1 u_1 - c_2 u_2 - u_3). \end{cases}$$

Using the moving coordinate $z = x - st$, (1) is written as

$$(u_i)_t = (u_i)_{zz} + s(u_i)_z + u_i g_i(u), \; z \in \mathbb{R}, \; t > 0, \; i = 1, 2, 3. \qquad (14)$$

Theorem 6 (3 weak competitors)

Assume, in addition to (9), that either $a_2b_3c_1 = a_3b_1c_2$ *or*

$$a_2 + b_1 + a_3 + c_1 \leq 2,\ a_2 + b_1 + b_3 + c_2 \leq 2,\ a_3 + c_1 + b_3 + c_2 \leq 2.$$

Let Φ *be a positive traveling wave of* (1) *connecting* $E_+ \in \{(0,0,1),(0,v_c,w_c)\}$ *and* $E_- = E_*$ *with wave speed* $s \geq 2\sqrt{R}$*, where* $R := \max\{r_1, r_2, r_3\}$*. Then* $\{s, \Phi\}$ *is stable in the sense that* $u(z,t) \to \Phi(z)$ *as* $t \to \infty$ *locally uniformly for* $z \in \mathbb{R}$ *for any solution* u *of* (14) *with initial data* u_0 *at* $t = 0$ *satisfying* $e^{-\lambda z}\mathcal{K}[u_0] \in L^1(\mathbb{R})$*, where* $\lambda = \lambda(s;R)$ *is defined in* (12) *and the positive constants* $\{\sigma_1, \sigma_2, \sigma_3\}$ *in* (11) *are chosen so that* (13) *holds.*

Theorem 7 (2 weak vs 1 strong competitors)

Assume, in addition to

$$a_2 + a_3 < 1,\ b_3, c_2 < 1,\ b_1 \geq 1/a_2,\ c_1 \geq 1/a_3, \tag{15}$$

that

$$a_2 b_1 = 1 = a_3 c_1,\ 2a_2 a_3 = a_2^2 b_3 + a_3^2 c_2. \tag{16}$$

Let Φ *be a positive traveling wave of* (1) *connecting* $E_+ \in \{(0,0,1),(0,v_c,w_c)\}$ *and* $E_- = E_u$ *with wave speed* $s \geq 2\sqrt{R}$, *where* $R := \max\{r_1, r_2, r_3\}$. *Then* $\{s, \Phi\}$ *is stable in the sense described in Theorem 6.*

- At the stable tail of traveling wave, i.e., $z = -\infty$, the perturbation is allowed to be arbitrarily large due to condition $e^{-\lambda z}\mathcal{K}[u_0] \in L^1(\mathbb{R})$. Note that $\lambda < 0$.
- However, at the unstable tail ($z = \infty$), the perturbation can only be made with decay rate faster than $e^{\lambda z}$.
  Note that the exponent λ is a function of the wave speed s.
- This is a typical phenomenon in the stability of monostable waves in many reaction-diffusion systems, including their discrete analogues.
- Note that $R > r_1(1 - a_3)$ and $R > r_1\beta$. The stability of wave with speed $s < 2\sqrt{R}$ is still left open.

§3-2. Proofs for stability

Let $\Psi(z,t) := \mathcal{K}[u(\cdot,t)](z)$, $z \in \mathbb{R}$, $t > 0$. Since

$$\max_{1\le i\le 3} \sup_{z\in\mathbb{R}}\{g_i(\Phi(z))\} \le \max\{r_1, r_2, r_3\} := R,$$

we can check that Ψ satisfies

$$\Psi_t \le \Psi_{zz} + s\Psi_z + R\Psi,\ \ z \in \mathbb{R},\ t > 0, \qquad (17)$$

if (13) holds for some $\{\sigma_1, \sigma_2, \sigma_3\}$.
From (17) and $e^{-\lambda z}\mathcal{K}[u_0] \in L^1(\mathbb{R})$, by comparison, Theorems 6 and 7 are proved. □

Check the condition (13): $I \geq 0$

Let $u := (u_1, u_2, u_3)$, $v := (v_1, v_2, v_3)$ and

$$X := u_1 - v_1,\ Y := u_2 - v_2,\ Z := u_3 - v_3.$$

Then I is computed as

$$\begin{aligned} I &= \sigma_1 r_1 X^2 + \sigma_2 r_2 Y^2 + \sigma_3 r_3 Z^2 + (\sigma_1 r_1 a_2 + \sigma_2 r_2 b_1) XY \\ &\quad + (\sigma_2 r_2 b_3 + \sigma_3 r_3 c_2) YZ + (\sigma_1 r_1 a_3 + \sigma_3 r_3 c_1) XZ. \end{aligned}$$

Three weak competitors

Case 1. $a_2b_3c_1 = a_3b_1c_2$: choose

$$\sigma_1 = \frac{1}{r_1},\ \sigma_2 = \frac{a_2}{r_2b_1},\ \sigma_3 = \frac{a_3}{r_3c_1},$$

then we can write

$$\begin{aligned} I &= (1-a_2-a_3)X^2 + \frac{a_2}{b_1}(1-b_1-b_3)Y^2 + \frac{a_3}{c_1}(1-c_1-c_2)Z^2 \\ &\quad + a_2(X+Y)^2 + a_3(X+Z)^2 + \frac{a_2b_3}{b_1}(Y+Z)^2. \end{aligned}$$

Case 2. condition

$$a_2+b_1+a_3+c_1 \le 2,\ a_2+b_1+b_3+c_2 \le 2,\ a_3+c_1+b_3+c_2 \le 2. \quad (18)$$

Choosing $\sigma_i = 1/r_i$, $i = 1, 2, 3$, we obtain that

$$\begin{aligned} I &= X^2 + Y^2 + Z^2 + (a_2+b_1)XY + (a_3+c_1)XZ + (b_3+c_2)YZ, \\ &= [X, Y, Z] \begin{bmatrix} 1 & (a_2+b_1)/2 & (a_3+c_1)/2 \\ (a_2+b_1)/2 & 1 & (b_3+c_2)/2 \\ (a_3+c_1)/2 & (b_3+c_2)/2 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}. \end{aligned}$$

Then $I \ge 0$ under condition (18), by Gerschgorin's Theorem.

Two weak and one strong competitors

Choosing

$$\sigma_1 = \frac{1}{r_1},\ \sigma_2 = \frac{a_2}{b_1 r_2},\ \sigma_3 = \frac{a_3}{c_1 r_3},$$

from (16) and (15), we can write

$$I = [X, Y, Z] \begin{bmatrix} 1 & a_2 & a_3 \\ a_2 & a_2^2 & a_2 a_3 \\ a_3 & a_2 a_3 & a_3^2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

Since B has the eigenvalues $\{0, 0, 1 + a_2^2 + a_3^2\}$ and so B is symmetric positive semi-definite, we obtain $I \geq 0$.

In fact, we have $I = (X + a_2 Y + a_3 Z)^2$. Q: any other conditions?

Thank you for your listening!

"Stability of single transition layer solutions in mass-conserving reaction-diffusion systems with bistable nonlinearity"

Hideo Ikeda（University of Toyama, Japan）

Mass-conserving reaction-diffusion systems with bistable nonlinearity are considered under general assumptions, which are useful models for studying cell polarity formation, whose process is key in cell division and differentiation. The existence of stationary solutions with a single internal transition layer is shown by using the analytical singular perturbation theory. Moreover, a stability criterion for the stationary solutions is provided by calculating the Evans function. This is a joint work with Masataka Kuwamura of Kobe University.

Stability of single transition layer solutions in mass-conserving reaction-diffusion systems with bistable nonlinearity

University of Toyama
Department of Mathematics
Hideo Ikeda

Jointly with
Masataka Kuwamura
Kobe University

ICMMA2023
Reaction-diffusion systems: from the past to the future
dedicated to the memory of Prof. Masayasu Mimura.
October 31 – November 2, 2023, on the Nakano
Campus of Meiji University, Tokyo.

In Memoriam of Professor Mayan Mimura from my youth

・First Encounter

A camp-style C&A summer seminar in 1978. This seminar was a study group where participants related to computation and analysis from companies and universities gathered for three days and two nights, eating and sleeping together.
(2泊3日の合宿セミナー)

My first impression of Mayan
He was serious and strict during the daytime seminar.
However, at night, he was very friendly and had a sense of humor.
I liked him at night.
This is a scene from that night.

Night scene in a Japanese-style room

- Opportunity to start the analytical singular perturbation method (1980)

He introduced the analytical singular perturbation method to Japan as a souvenir when he came back from Oxford University. I was very interested in the method of constructing an approximate solution for each domain and then successfully combining them at the end. The content was an introduction of a paper by P.C. Fife

- Mayan's Intensive Lecture for Undergraduate Students at my university (1982)

The subject matter was something I had never heard before in a mathematics class.
The title was "A Model of Pulse Propagation on Neuronal Axons (Hodgkin-Huxley Equation).
------ Differential equations that won the Nobel Prize in Physiology and Medicine in 1963

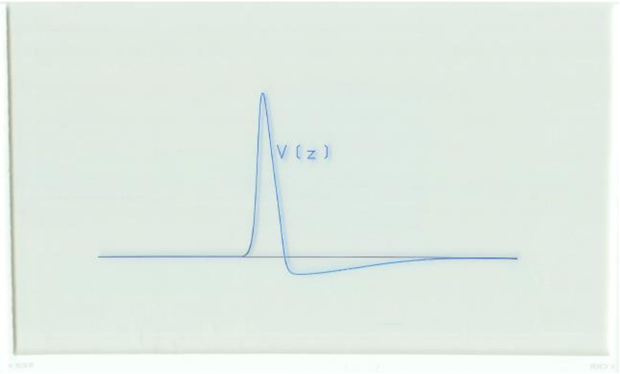
I wondered for a moment if this was really mathematics. I thought for a moment, but as I listened, differential equations naturally came to my face. At that time, I felt freshness and interest in this kind of mathematics.

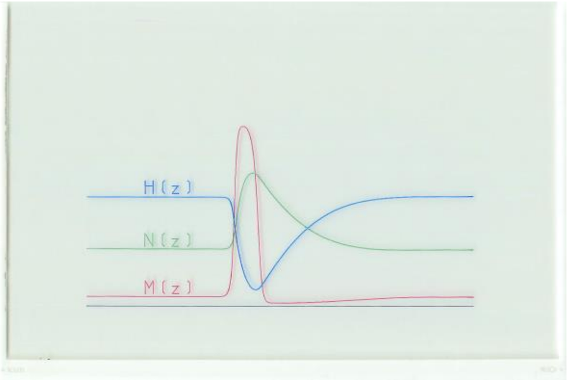
- Start of joint research with Mayan and T. Tsujikawa (1983)

Hodgkin-Huxley Equation

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} - \bar{g}_K n^4 (v - V_K) \\ \qquad - \bar{g}_{Na} m^3 h (v - V_{Na}) - \bar{g}_\ell (v - V_\ell) \\ \dfrac{\partial m}{\partial t} = \gamma_m(v)(m_\infty(v) - m) \\ \dfrac{\partial n}{\partial t} = \gamma_n(v)(n_\infty(v) - n) \\ \dfrac{\partial h}{\partial t} = \gamma_h(v)(h_\infty(v) - h) \end{cases} \quad , t > 0, \; x \in \mathbf{R}$$

Calculated by NEC PC-9801 (MS-DOS) + BASIC





H. Ikeda, M. Mimura, T. Tsujikawa, Slow traveling wave solutions to the Hodgkin–Huxley equations. Recent topics in nonlinear PDE, III (Tokyo, 1986), 1–73, North–Holland Math. Stud., 148, Lecture Notes Numer. Appl. Anal., 9, North–Holland, Amsterdam, 1987.

H. Ikeda, M. Mimura, T. Tsujikawa, Singular perturbation approach to traveling wave solutions of the Hodgkin–Huxley equations and its application to stability problems. Japan J. Appl. Math. 6 (1989), no.1, 1–66.

T. Tsujikawa, T. Nagai, M. Mimura, R. Kobayashi, H. Ikeda, Stability properties of traveling pulse solutions of the higher–dimensional FitzHugh–Nagumo equations. Japan J. Appl. Math. 6 (1989), no.3, 341–366.

- One year research life at Hiroshima University as an in-country student in 1986
  ---- with support from the Ministry of Education (MEXT)

At that time, Mayan's wife was returning to her parents' home for childbirth. Therefore, Mayan had no choice but to go home early, so he stayed with me one-on-one until nighttime. However, we did not talk about research all the time. He was absorbed in thinking of problems that he thought would be interesting to solve. He would often say to me, "How about this problem? But you are the one who will solve this problem".
Thanks to him, I was able to write the following five papers.

H. Ikeda, On the asymptotic solutions for a weakly coupled elliptic boundary value problem with a small parameter. Hiroshima Math. J. 16 (1986), no. 2, 227–250.

H. Ikeda, M. Mimura, Y. Nishiura, Global bifurcation phenomena of travelling wave solutions for some bistable reaction–diffusion systems. Nonlinear Anal. 13 (1989), no.5, 507–526.

H. Ikeda, M. Mimura, Wave-blocking phenomena in bistable reaction–diffusion systems. SIAM J. Appl. Math. 49 (1989), no.2, 515–538.

Y. Nishiura, M. Mimura, H. Ikeda, H. Fujii, Singular limit analysis of stability of traveling wave solutions in bistable reaction–diffusion systems. SIAM J. Math. Anal. 21 (1990), no.1, 85–122.

H. Ikeda, M. Mimura, Stability analysis of stationary solutions of bistable reaction–variable diffusion systems. SIAM J. Math. Anal. 22 (1991), no. 6, 1651–1678.

One-day trip to Miyajima with Mayan's family

BBQ + wine party on the roof of Mayan's apartment

Thanks to these, I received a Doctor of Science degree from Hiroshima University in 1989

- Deep acknowledgment to Mayan

I would like to take this opportunity to express my sincere gratitude to Mayan for giving me a start in my research life and for his words of encouragement every time we met.

Reaction-diffusion systems are useful models for understanding the mechanism of appearance of non-uniform patterns in various fields.

$$(1) \quad \begin{cases} u_t = \varepsilon^2 u_{xx} + f(u,v), & \\ v_t = D v_{xx} - f(u,v), & (t,x) \in (0,\infty) \times (0,1) \\ (u_x, v_x)(t,0) = (0,0) = (u_x, v_x)(t,1), \ 0 < t < \infty, \end{cases}$$

where ε and D are positive constants satisfying $0 < \varepsilon \ll D$.

We note that (1) is a mass-conserving reaction-diffusion system because

$$(2) \qquad \xi := \int_0^1 \{u(x,0) + v(x,0)\}\, dx = \int_0^1 \{u(x,t) + v(x,t)\}\, dx$$

holds for any (smooth) solutions.

Y. Mori, A. Jilkine and L. Edelstein-Keshet, Wave-pinning and cell polarity from a bistable reaction-diffusion system, Biophys. J. **94** (2008) 3684–3697.

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + f(u,v), \tag{2.1a}$$

$$\frac{\partial v}{\partial t} = D_v \frac{\partial^2 v}{\partial x^2} - f(u,v), \tag{2.1b}$$

where $f(u,v)$ is the rate of interconversion of v to u, and the rates of diffusion satisfy $D_u \ll D_v$, reflecting the fact that the membrane bound species u diffuses much more slowly than the cytosolic species v. The boundary conditions are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0, \quad x = 0, L. \tag{2.1c}$$

It is clear that system (2.1) leads to mass conservation, i.e., that

$$\int_\Omega (u+v)dx = K_{\text{total}} = \text{constant}. \tag{2.2}$$

$$f(u,v) = (\text{activation rate}) \cdot v - (\text{inactivation rate}) \cdot u = \eta \left(\delta + \frac{\gamma u^2}{m^2 + u^2} \right) v - \eta u,$$

The model is based on a caricature of Rho proteins:

(1) The protein has an active (GTP-bound) form (u) and an inactive (GDP-bound) form (v).

(2) The active forms are found only on the cell membrane; those in the fluid interior of the cell (cytosol) are inactive.

(3) There is a 100-fold difference between rates of diffusion of cytosolic vs. membrane bound proteins.

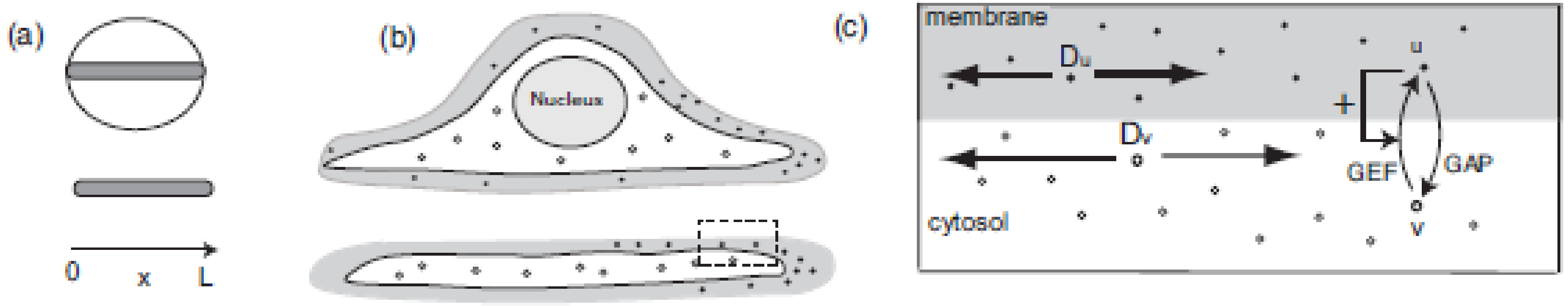
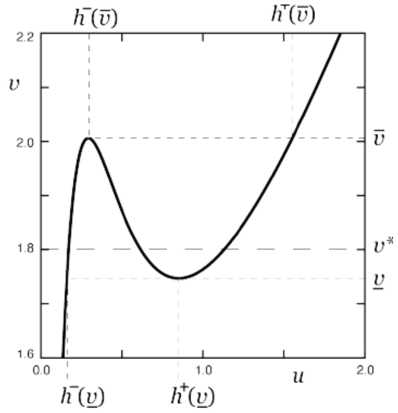


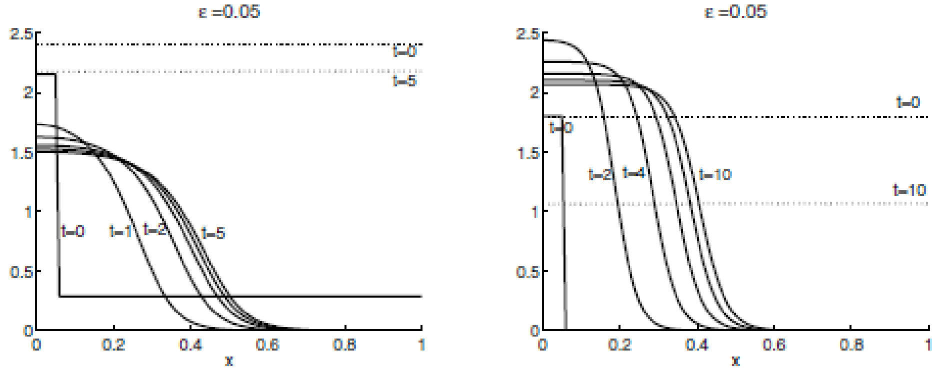
FIG. 2.1. (a) *Our 1D model represents a strip across a cell diameter ($L \approx 10\mu m$), shown top-down and side view.* (b) *Side view of a cell (top) showing membrane (shaded) and cytosol (white) and a cell fragment (bottom) $\approx 0.1\mu m$ thick; see [40]. Active ($u(x,t)$, black dots) and inactive ($v(x,t)$, white dots) proteins redistribute along this axis during polarization.* (c) *Enlarged rectangle from* (b) *showing exchange between membrane and cytosol ($u \leftrightarrow v$), unequal rates of diffusion, inactivation by GAPs, and activation by GEFs with positive feedback (+ arrow) (schematic not drawn to scale).*

Y. Mori, A. Jilkine, L. Edelstein-Keshet, Asymptotic and bifurcation analysis of wave-pinning in a reaction-diffusion model for cell polarization, SIAM J. Appl. Math **71** (2011), 1401-1427.

$$f(u,v) = \frac{u^2 v}{1+u^2} - u. \tag{2.13}$$

$$f(u,v) = u(1-u)(u-1-v), \tag{2.14}$$





(a) (b)

FIG. 2.2. *Wave-pinning behavior for the RD model (2.6) with $\epsilon = 0.05$, $D = 1$. (a) Hill function reaction kinetics (2.13) with $\delta = 0$, $\gamma = 1$, $m = 1$, $K = 2.8$. (b) Cubic reaction kinetics (2.14) and $K = 1.9$. Solutions to u (solid) and v (dashed) are shown at the indicated times. The wave is initiated as the square pulse in u at $t = 0$.*

The wave-pinning is a phenomenon that a wave of activation of the species is initiated at one end of the domain, moves into the domain, decelerates, and eventually stops inside the domain, forming a stationary front (cell polarization).

Y. Mori, A. Jilkine, L. Edelstein-Keshet, Asymptotic and bifurcation analysis of wave-pinning in a reaction-diffusion model for cell polarization, SIAM J. Appl. Math **71** (2011), 1401-1427.

They claimed in this paper that the system (1) has a stable stationary solution with a single internal transition layer under certain conditions by using a formal asymptotic analysis, numerical computations and a perturbative argument against the background of cell biology.

The aim of my talk is to show rigorusly that the system (1) has a stable stationary solution with a single internal transition layer under more general nonlinearity f.

Assumption.

(A1) The ODE $\dot{u} = f(u, v)$ is bistable in u for each fixed $v \in I = (\underline{v}, \overline{v})$. That is, $f(u, v) = 0$ has exactly three roots $h^-(v) < h^0(v) < h^+(v)$ for each $v \in I$ satisfying

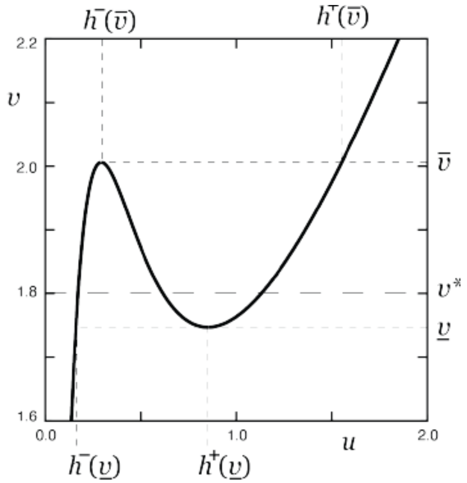
$$f_u(h^{\pm}(v), v) < 0 \quad \text{and} \quad f_u(h^0(v), v) > 0.$$

(A2) The function

$$J(v) := \int_{h^-(v)}^{h^+(v)} f(u, v)du \quad (v \in I)$$

has an isolated zero at $v = v^* \in I$ such that

$$J'(v^*) = \int_{h^-(v^*)}^{h^+(v^*)} f_v(u, v^*)du \neq 0.$$



(A3)

$$f_u(h^{\pm}(v), v) < f_v(h^{\pm}(v), v) \quad (v \in I).$$

(A4) The conserved mass ξ satisfies the following inequality:

$$h^-(v^*) + v^* < \xi < h^+(v^*) + v^*.$$

Existence of single transition layer solutions

$$(3)\quad \begin{cases} \varepsilon^2 u_{xx} + f(u,v) = 0, & \\ D v_{xx} - f(u,v) = 0, & x \in (0,1) \\ (u_x, v_x)(0) = (0,0) = (u_x, v_x)(1) & \end{cases}$$

satisfying

$$(4)\qquad \xi = \int_0^1 \{u(x) + v(x)\}\, dx$$

for a given constant ξ in (A4).

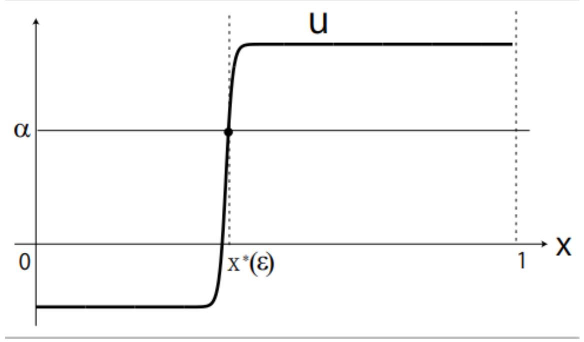
$\Longrightarrow \varepsilon^2 u + Dv = C(\varepsilon)$

Substituting $v = (C(\varepsilon) - \varepsilon^2 u)/D$, we have a single equation for u

$$\begin{cases} \varepsilon^2 u_{xx} + f(u, (C(\varepsilon) - \varepsilon^2 u)/D) = 0, & x \in (0,1) \\ u_x(0) = 0 = u_x(1), & \end{cases}$$

and

$$\xi = \frac{C(\varepsilon)}{D} + \left(1 - \frac{\varepsilon^2}{D}\right) \int_0^1 u(x)dx,$$



By using the singular perturbation technique, we can obtain $u(x;\varepsilon)$, $C(\varepsilon)$ and $x^*(\varepsilon)$ satisfying $u(x^*(\varepsilon);\varepsilon) = \alpha$.

Stability analysis of the single transition layer solutions

When we apply the SLEP method ➔

$$D\psi_{xx} + \frac{f_u^\varepsilon f_v^\varepsilon}{f_u^\varepsilon - \lambda}\psi - f_v^\varepsilon \psi - \lambda\psi = o(1) \quad \text{as} \quad \varepsilon \to 0 \text{ for } \psi \in H^2[0,1] \cap H_N^1[0,1].$$

$$\langle \frac{f_u^\varepsilon f_v^\varepsilon}{f_u^\varepsilon - \lambda}\psi - f_v^\varepsilon \psi - \lambda\psi, \psi \rangle \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{for a crirtcal } \lambda = o(1) \quad (||\psi||_{L^2} = 1).$$

The Lax-Milgram theorem cannot be applied to the solvability of the SLEP equation.
This shortcoming seems to be common to singular perturbation problems for mass-conserving reaction-diffusion systems such as (1).

In this talk, we consider the full system (3).

$$(3) \qquad \begin{cases} \varepsilon^2 u_{xx} + f(u,v) = 0, \\ D v_{xx} - f(u,v) = 0, \\ (u_x, v_x)(0) = (0,0) = (u_x, v_x)(1) \end{cases} \quad x \in (0,1)$$

satisfying

$$(4) \qquad \xi = \int_0^1 \{u(x) + v(x)\}\, dx$$

for a given constant ξ in (A4).

This is a reason why it is convenient to show stability property

------- to calculate the Evans function .

Divide $[0,1]$ into two subintervals $[0, x^*(\varepsilon)]$ and $[x^*(\varepsilon), 1]$, and consider the following two boundary value problems:

$$(5) \quad \begin{cases} \varepsilon^2 u_{xx} + f(u,v) = 0, & x \in (0, x^*(\varepsilon)) \\ D v_{xx} - f(u,v) = 0, & \\ (u_x, v_x)(0) = (0,0), \ (u,v)(x^*(\varepsilon)) = (\alpha, \beta(\varepsilon)) & \end{cases}$$

and

$$(6) \quad \begin{cases} \varepsilon^2 u_{xx} + f(u,v) = 0, & x \in (x^*(\varepsilon), 1) \\ D v_{xx} - f(u,v) = 0, & \\ (u,v)(x^*(\varepsilon)) = (\alpha, \beta(\varepsilon)), \ (u_x, v_x)(1) = (0,0), & \end{cases}$$

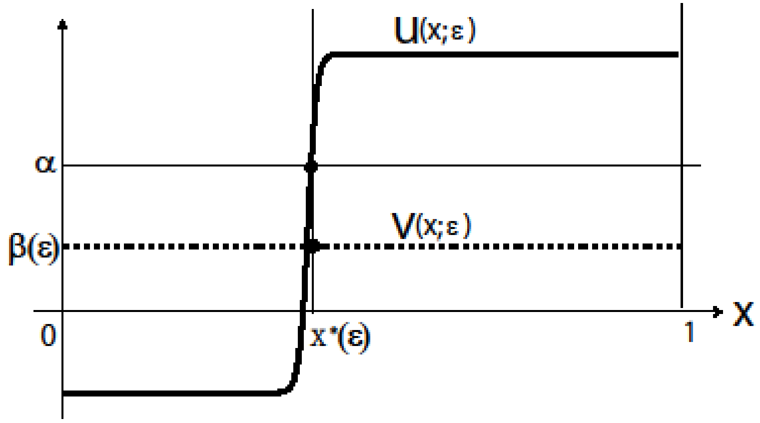
where α is an arbitrary fixed constant satisfying $h^-(v^*) < \alpha < h^+(v^*)$ and $\beta(\varepsilon)$ is determined by $v(x^*(\varepsilon)) = \beta(\varepsilon)$. First, we suppose that

$$x^*(\varepsilon) = x_0 + \varepsilon x_1$$

and

$$\beta(\varepsilon) = \beta_0 + \varepsilon \beta_1.$$

are arbitrarily given.



By using the singuler perturbation method, we can construct solutions $(u^{\pm}, v^{\pm})(x;\varepsilon)$ of (5) and (6), respectively, which have uniformly $O(\varepsilon)$-approximations.

Next, we match both solutions $(u^{\pm}, v^{\pm})(x;\varepsilon)$ in C^1–sense at $x = x^*(\varepsilon)$, and use the constrained condition $\xi = \int_0^1 \{u(x;\varepsilon) + v(x;\varepsilon)\}\, dx$,

from which $x^*(\varepsilon) = x_0 + \varepsilon x_1(\varepsilon)$ and $\beta(\varepsilon) = \beta_0 + \varepsilon\beta_1(\varepsilon)$ are determined uniquely.

($x_0 = (v^* + h^+(v^*) - \xi)/(h^+(v^*) - h^-(v^*))$, $\beta_0 = v^*$)

Remark $\varepsilon^2 u + Dv = C(\varepsilon)$.

→ Theorem 1 (Existence). There exists a solution $(u, v)(x;\varepsilon)$ satisfying

$$(3) \quad \begin{cases} \varepsilon^2 u_{xx} + f(u,v) = 0, & \\ Dv_{xx} - f(u,v) = 0, & x \in (0,1) \\ (u_x, v_x)(0) = (0,0) = (u_x, v_x)(1) & \end{cases}$$

and

$$(4) \qquad \xi = \int_0^1 \{u(x;\varepsilon) + v(x;\varepsilon)\}\, dx.$$

Stability analysis of the single transition layer solutions

We consider the linearized eigenvalue problem of (1)

$$(7) \quad \mathcal{L}^{\varepsilon}\begin{bmatrix} p \\ q \end{bmatrix} := \begin{bmatrix} \varepsilon^2 \dfrac{d^2}{dx^2} + f_u^{\varepsilon} & f_v^{\varepsilon} \\ -f_u^{\varepsilon} & D\dfrac{d^2}{dx^2} - f_v^{\varepsilon} \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = \lambda \begin{bmatrix} p \\ q \end{bmatrix},$$

under the Neumann boundary condition, where $f_u^{\varepsilon} := f_u(u(x;\varepsilon), v(x;\varepsilon))$, $f_v^{\varepsilon} := f_v(u(x;\varepsilon), v(x;\varepsilon))$ and $\lambda \in \mathbb{C}$. The underlying space for (7) can be taken as $BC[0,1] \times BC[0,1]$ with

$$\mathcal{D}(\mathcal{L}^{\varepsilon}) := \{(p,q) \in \mathring{C}^2[0,1] \times \mathring{C}^2[0,1] \mid \int_0^1 (p+q)dx = 0\}$$

by virtue of (2). We note that for $(p,q) \in \mathring{C}^2[0,1] \times \mathring{C}^2[0,1]$ satisfying (7), the condition

$$\lambda \int_0^1 (p+q)dx = 0$$

always holds by integrating the equations with respect to p and q in (7) on the interval $[0,1]$ under the Neumann boundary conditions. This fact implies that $(p,q) \in \mathcal{D}(\mathcal{L}^{\varepsilon})$ if $(p,q) \in \mathring{C}^2[0,1] \times \mathring{C}^2[0,1]$ satisfies (7) for $\lambda \neq 0$.

$$\mathring{C}^2[0,1] := \{u \in C^2[0,1] \mid u_x(0) = 0,\ u_x(1) = 0\}$$

The equation (7) can be rewritten equivalently as

$$(8)\qquad \begin{cases} \dfrac{d}{dx}\bar{V} = A(x;\varepsilon;\lambda)\bar{V}, \quad x \in (0,1) \\ (p_x, q_x)(0) = (0,0), \quad (p_x, p_x)(1) = (0,0) \end{cases}$$

for $\bar{V} = \bar{V}(x;\varepsilon;\lambda) := (p, \varepsilon p_x, q, q_x)\,(x;\varepsilon;\lambda)$, where $A(x;\varepsilon;\lambda)$ is defined by

$$A(x;\varepsilon;\lambda) := \begin{bmatrix} 0 & 1/\varepsilon & 0 & 0 \\ (\lambda - f_u^\varepsilon)/\varepsilon & 0 & -f_v^\varepsilon/\varepsilon & 0 \\ 0 & 0 & 0 & 1 \\ f_u^\varepsilon/D & 0 & (\lambda + f_v^\varepsilon)/D & 0 \end{bmatrix}.$$

Similarly to the construction of the single transition layer solution, we can solve the following problems with suitable boundary conditions:

$$(9)\quad \begin{cases} \varepsilon^2 p_{xx} + f_u^\varepsilon p + f_v^\varepsilon q = \lambda p, \\ D q_{xx} - f_u^\varepsilon p - f_v^\varepsilon q = \lambda q, \end{cases} \quad x \in (0, x^*(\varepsilon)) \\ \quad (p_x, q_x)(0) = (0,0), \ (p,q)(x^*(\varepsilon)) = (a,b).$$

and

$$(10)\quad \begin{cases} \varepsilon^2 p_{xx} + f_u^\varepsilon p + f_v^\varepsilon q = \lambda p, \\ D q_{xx} - f_u^\varepsilon p - f_v^\varepsilon q = \lambda q, \end{cases} \quad x \in (x^*(\varepsilon), 1) \\ \quad (p,q)(x^*(\varepsilon)) = (a,b), \ (p_x, q_x)(1) = (0,0),$$

where a, b are given real numbers. For any $\lambda \in \mathbb{C}$, let $(p^-, q^-)(x;\varepsilon;\lambda;a,b)$ and $(p^+, q^+)(x;\varepsilon;\lambda;a,b)$ be solutions of (9) and (10), respectively.

Then, any solution $\bar{V}(x;\varepsilon;\lambda)$ of (8) on $[0, x^*(\varepsilon)]$ is represented as a linear combination of two independent solutions

$$\bar{V}_1(x;\varepsilon;\lambda) := \begin{bmatrix} p^-(x;\varepsilon;\lambda;1,0) \\ \varepsilon p_x^-(x;\varepsilon;\lambda;1,0) \\ q^-(x;\varepsilon;\lambda;1,0) \\ q_x^-(x;\varepsilon;\lambda;1,0) \end{bmatrix}, \quad \bar{V}_2(x;\varepsilon;\lambda) := \begin{bmatrix} p^-(x;\varepsilon;\lambda;0,1) \\ \varepsilon p_x^-(x;\varepsilon;\lambda;0,1) \\ q^-(x;\varepsilon;\lambda;0,1) \\ q_x^-(x;\varepsilon;\lambda;0,1) \end{bmatrix}.$$

Similarly, any solution of (8) on $[x^*(\varepsilon), 1]$ is represented as a linear combination of two independent solutions

$$\bar{V}_3(x;\varepsilon;\lambda) := \begin{bmatrix} p^+(x;\varepsilon;\lambda;1,0) \\ \varepsilon p_x^+(x;\varepsilon;\lambda;1,0) \\ q^+(x;\varepsilon;\lambda;1,0) \\ q_x^+(x;\varepsilon;\lambda;1,0) \end{bmatrix}, \quad \bar{V}_4(x;\varepsilon;\lambda) := \begin{bmatrix} p^+(x;\varepsilon;\lambda;0,1) \\ \varepsilon p_x^+(x;\varepsilon;\lambda;0,1) \\ q^+(x;\varepsilon;\lambda;0,1) \\ q_x^+(x;\varepsilon;\lambda;0,1) \end{bmatrix}.$$

Since the coefficient matrix $A(x;\varepsilon;\lambda)$ of (8) depends analytically on λ, we can consider, without loss of generality, that $\bar{V}_i(x;\varepsilon;\lambda)$ $(i = 1, 2, 3, 4)$ also depend analytically on λ

Let $\bar{V}(x;\varepsilon;\lambda)$ be a nontrivial solutions of (8) for some $\lambda \in \mathbb{C}$. Then, there exist constants α_i $(i = 1,2,3,4)$ satisfying $\sum_{i=1}^{4}|\alpha_i| \neq 0$ such that $\bar{V}(x;\varepsilon;\lambda)$ must be represented as

$$\bar{V}(x;\varepsilon;\lambda) = \begin{cases} \alpha_1\bar{V}_1(x;\varepsilon;\lambda) + \alpha_2\bar{V}_2(x;\varepsilon;\lambda), & x \in [0, x^*(\varepsilon)] \\ \alpha_3\bar{V}_3(x;\varepsilon;\lambda) + \alpha_4\bar{V}_4(x;\varepsilon;\lambda), & x \in [x^*(\varepsilon), 1], \end{cases}$$

which implies that the relation

$$\alpha_1\bar{V}_1(x^*(\varepsilon);\varepsilon;\lambda) + \alpha_2\bar{V}_2(x^*(\varepsilon);\varepsilon;\lambda) = \alpha_3\bar{V}_3(x^*(\varepsilon);\varepsilon;\lambda) + \alpha_4\bar{V}_4(x^*(\varepsilon);\varepsilon;\lambda).$$

holds at $x = x^*(\varepsilon)$; four vectors $\bar{V}_i(x^*(\varepsilon);\varepsilon;\lambda)$ $(i = 1,2,3,4)$ are linearly dependent. Defining

$$g(\varepsilon;\lambda) := \det[\bar{V}_1(x^*(\varepsilon);\varepsilon;\lambda),\ \bar{V}_2(x^*(\varepsilon);\varepsilon;\lambda), \bar{V}_3(x^*(\varepsilon);\varepsilon;\lambda), \bar{V}_4(x^*(\varepsilon);\varepsilon;\lambda)],$$

we find that $g(\varepsilon;\lambda)$ is an analytic function of $\lambda \in \mathbb{C}$ and have the next lemma:

Lemma 1 Let $\lambda \neq 0$. Then, $\lambda \in \mathbb{C}$ is an eigenvalue of (7) if and only if $g(\varepsilon;\lambda) = 0$.

We call $g(\varepsilon;\lambda)$ the *Evans function* of the single transition layer solution, which enables us to investigate the distribution of eigenvalues of (7) in $\mathbb{C}$.

To calculate the Evans function, we have to construct functions $\bar{V}_i(x;\varepsilon;\lambda)$ $(i = 1,2,3,4)$ as we constructed a transition layer solution in the previous section. According to the dependency of $\lambda \in \mathbb{C}$ on ε, we must divide our argument into the following three cases:
(I) $\lambda = \lambda(\varepsilon) = O(\varepsilon)\ in\ \mathbb{C}$ as $\varepsilon \to 0$.

For the other two cases, we have $\lambda(\varepsilon)/\varepsilon \to \infty$ as $\varepsilon \to 0$. We find that there exists a positive and continuous real function $\omega(\varepsilon) \to \infty$ as $\varepsilon \to 0$ such that $\lambda(\varepsilon)$ is represented as

$$\lambda(\varepsilon) = \varepsilon\omega(\varepsilon)\hat{\lambda}(\varepsilon),$$

where $\hat{\lambda}(\varepsilon)$ satisfies $\hat{\lambda}(0) \neq 0$. Then, we consider two cases according to the magnitude of $\varepsilon\omega(\varepsilon)$ as follows:
(II) $\varepsilon\omega(\varepsilon) \to 0$ and $\omega(\varepsilon) \to \infty$ as $\varepsilon \to 0$;
(III) $\varepsilon\omega(\varepsilon) \to \omega_0$ as $\varepsilon \to 0$ for some positive constant ω_0.

Case (I) $\lambda = \lambda(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$

$$\tilde{g}(0;\kappa^*) = -\kappa^* \left[\kappa^* \int_{-\infty}^{\infty} (\dot{W}(z))^2 dz \int_0^1 \left(\frac{f_u^* - f_v^*}{f_u^*} \right) dx + (h^+(v^*) - h^-(v^*)) \int_{h^-(v^*)}^{h^+(v^*)} f_v(u, v^*) du \right] / \{D(\dot{W}(0))^2\},$$

where $W(z)$ is a solution of

$$\begin{cases} \ddot{W}(z) + f(W(z), v^*) = 0, \ z \in \mathbf{R}, \\ W(\pm\infty) = h^{\pm}(v^*), \ W(0) = \alpha; \end{cases}$$

Therefore, we find two solutions of $\tilde{g}(0;\kappa^*) = 0$ such that (i) $\kappa^* = 0$ or

(ii) $$\kappa^* = -\frac{(h^+(v^*) - h^-(v^*)) \int_{h^-(v^*)}^{h^+(v^*)} f_v(u, v^*) du}{\int_{-\infty}^{\infty} (\dot{W}(z))^2 dz \int_0^1 \left(\frac{f_u^* - f_v^*}{f_u^*} \right) dx} \neq 0.$$

For the case (I) $\kappa^* = 0$, $\int_0^1 \{p(x;\varepsilon) + q(x;\varepsilon)\}dx = 0 \Rightarrow \alpha_i = 0$ $(i = 1, 2, 3, 4)$, which implies that $(p, q)(x;\varepsilon) = (0, 0)$.

Thus, we obtain the following result:

Theorem 2(Case (I)) Assume that (A1)-(A4), and $\lambda = \lambda(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$. The eigenvalue problem (7) has only one eigenvalue

$$\lambda(\varepsilon) = -\frac{(h^+(v^*) - h^-(v^*)) \int_{h^-(v^*)}^{h^+(v^*)} f_v(u, v^*)du}{\int_{-\infty}^{\infty} (\dot{W}(z))^2 dz \int_0^1 \left(\frac{f_u^* - f_v^*}{f_u^*}\right) dx} \varepsilon + o(\varepsilon) \in \mathbb{C}$$

and the sign of the real part of $\lambda(\varepsilon)$ is determined by

$$sign\{\mathrm{Re}(\lambda(\varepsilon))\} \;=\; sign\left\{-\int_{h^-(v^*)}^{h^+(v^*)} f_v(u, v^*)du\right\} = sign\{-J'(v^*)\}.$$

Case (II) and Case (III)

$g(\varepsilon; \varepsilon\omega(\varepsilon)\hat{\lambda}(\varepsilon)) \neq 0$ for small $\varepsilon > 0$.

Theorem 3(Stability) Under the assumptions (A1)-(A4), for any fixed $d > 0$ the eigenvalue problem (7) has only one eigenvalue

$$\lambda(\varepsilon) = -\frac{(h^+(v^*) - h^-(v^*)) \int_{h^-(v^*)}^{h^+(v^*)} f_v(u, v^*) du}{\int_{-\infty}^{\infty} (\dot{W}(z))^2 dz \int_0^1 \left(\frac{f_u^* - f_v^*}{f_u^*} \right) dx} \varepsilon + o(\varepsilon)$$

in $\mathbb{C}_d$ and the sign of the real part of $\lambda(\varepsilon)$ is determined by

$$sign\{\mathrm{Re}(\lambda(\varepsilon))\} = sign\left\{ - \int_{h^-(v^*)}^{h^+(v^*)} f_v(u, v^*) du \right\} = sign\{-J'(v^*)\}.$$

Then, the single transition layer solution $(u, v)(x; \varepsilon)$ is stable when $J'(v^*) > 0$, conversely it is unstable when $J'(v^*) < 0$.

- Extension this method to reaction-diffusion system including a nonlocal terms.

The singular perturbation method is not good at dealing with nonlocal terms
because it is a method of constructing approximate solutions for each subdomain and
then successfully laminating them together at the end, but the method introduced here
can be extended to problems involving nonlocal terms.

Thanks for your kind attention

"Traveling wave solutions of combustion in a narrow channel"

Hirofumi Izuhara (Miyazaki University, Japan)

It is reported that combustion in a narrow channel shows a variety of char patterns depending on the airflow rate. In this talk, we consider a mathematical model which describes the combustion experiment, and numerically study the existence of traveling wave solution in one space dimension. In addition, we investigate the instability of its planar combustion wave which is the onset of the variety of char patterns.

TRAVELING WAVE SOLUTIONS OF COMBUSTION IN A NARROW CHANNEL

Hirofumi Izuhara University of Miyazaki

Joint work with

Kazunori Kuwana Tokyo University of Science

Tsuneyoshi Matsuoka Toyohashi University of Technology

ICMMA, Oct. 31st 2023, Meiji University

RESEARCH MOTIVE

Paper density is uniform, ignition is uniform and supply of oxygen is uniform.

Spreading of combustion is complicated even in a simple environment.

Can we understand the pattern forming mechanism?

Model aided understanding of combustion pattern formation

MATHEMATICAL MODEL

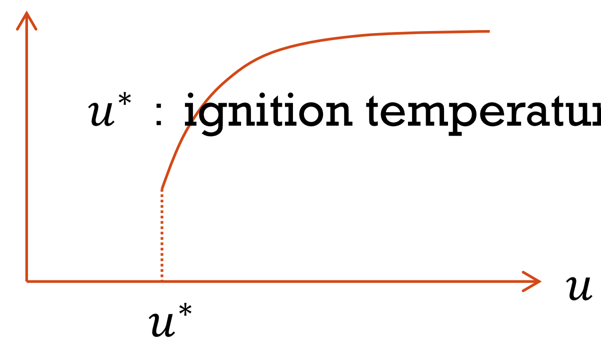
Fasano, Mimura and Primicerio proposed the following mathematical model:

$$\begin{aligned} u_t &= Le\Delta u + \phi\Lambda Pe u_x + \beta\gamma k(u)vw - a(u-\tilde{u}) \\ \phi v_t &= \Delta v + \phi Pe v_x - \gamma k(u)vw \\ w_t &= -H_w\gamma k(u)vw \end{aligned} \qquad t > 0, (x,y) \in \Omega$$

u : temperature, v : oxygen concentration, w : paper density

$k(u)$: Arrhenius law

$$k(u) = \begin{cases} \exp(-\theta/u) & u \geq u^* \\ 0 & u < u^* \end{cases}$$



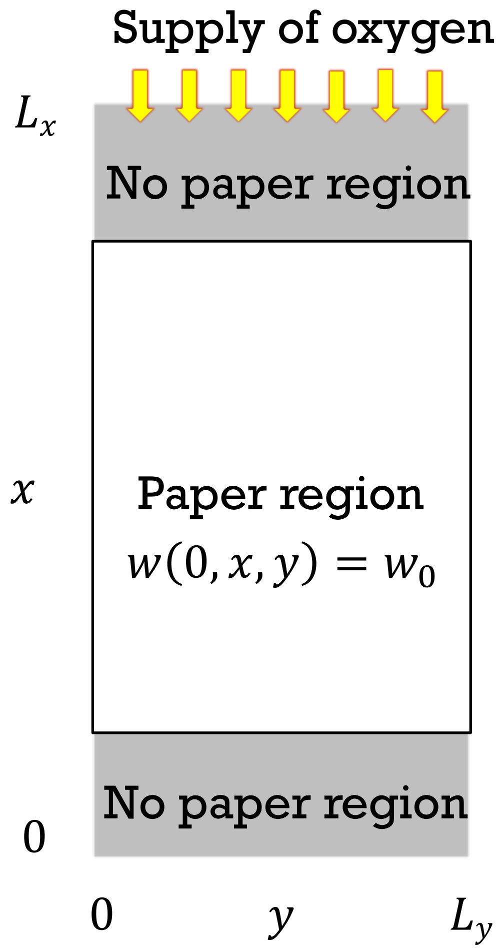
Fasano, Mimura & Primicerio \`09
Lu & Dong \`11
Ijioma, Izuhara, Mimura & Ogawa\`15

SETTING FOR SIMULATION

Parameter values:

$L_x = 300$
$L_y = 100$
$Le = 0.125$
$\phi = 1$
$\Lambda = 0$
$\beta = 10$
$\gamma = 140$
$a = 0.28$
$\tilde{u} = 0.02$
$\theta = 3$
$u^* = 0.08$
$H_w = 0.5$
$v_0 = 0.1$

$$\Omega = \{(x, y) | 0 < x < L_x, 0 < y < L_y\}$$



Initial conditions:

$u(0, x, y) = \tilde{u}$
$v(0, x, y) = v_0$

Boundary conditions:

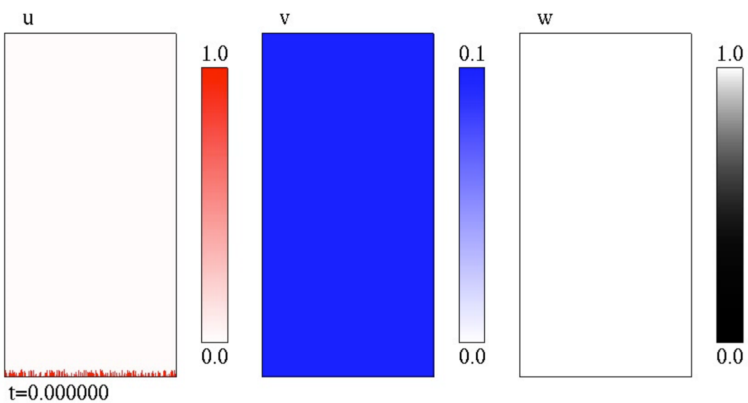
$v(t, Lx, y) = v_0$
The others are Neumann B.C.

Ignition:

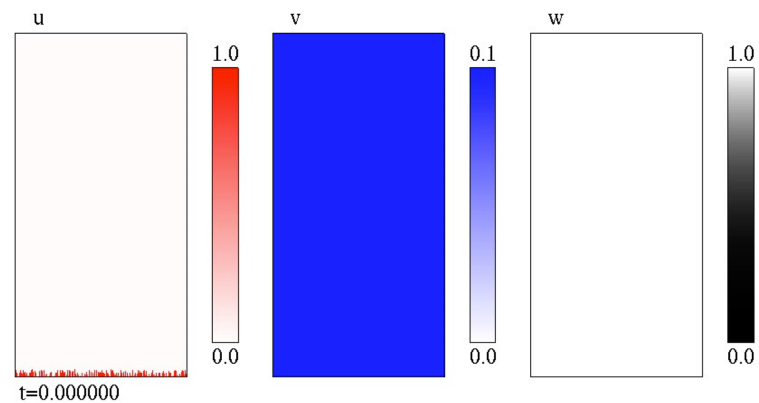
$u(0, x, y) = u^* + \varepsilon(x, y)$

NUMERICAL SIMULATIONS

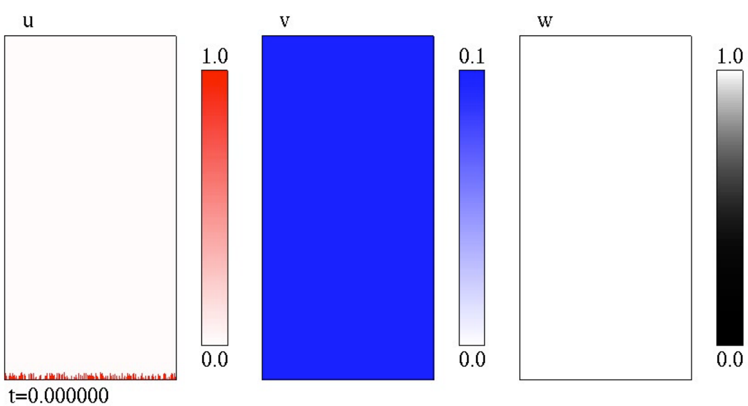
Ijioma, Izuhara & Mimura `17



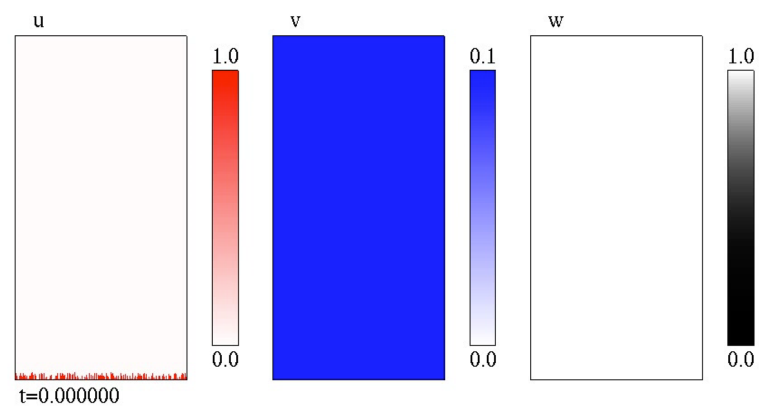
Pe=3



Pe=1.5



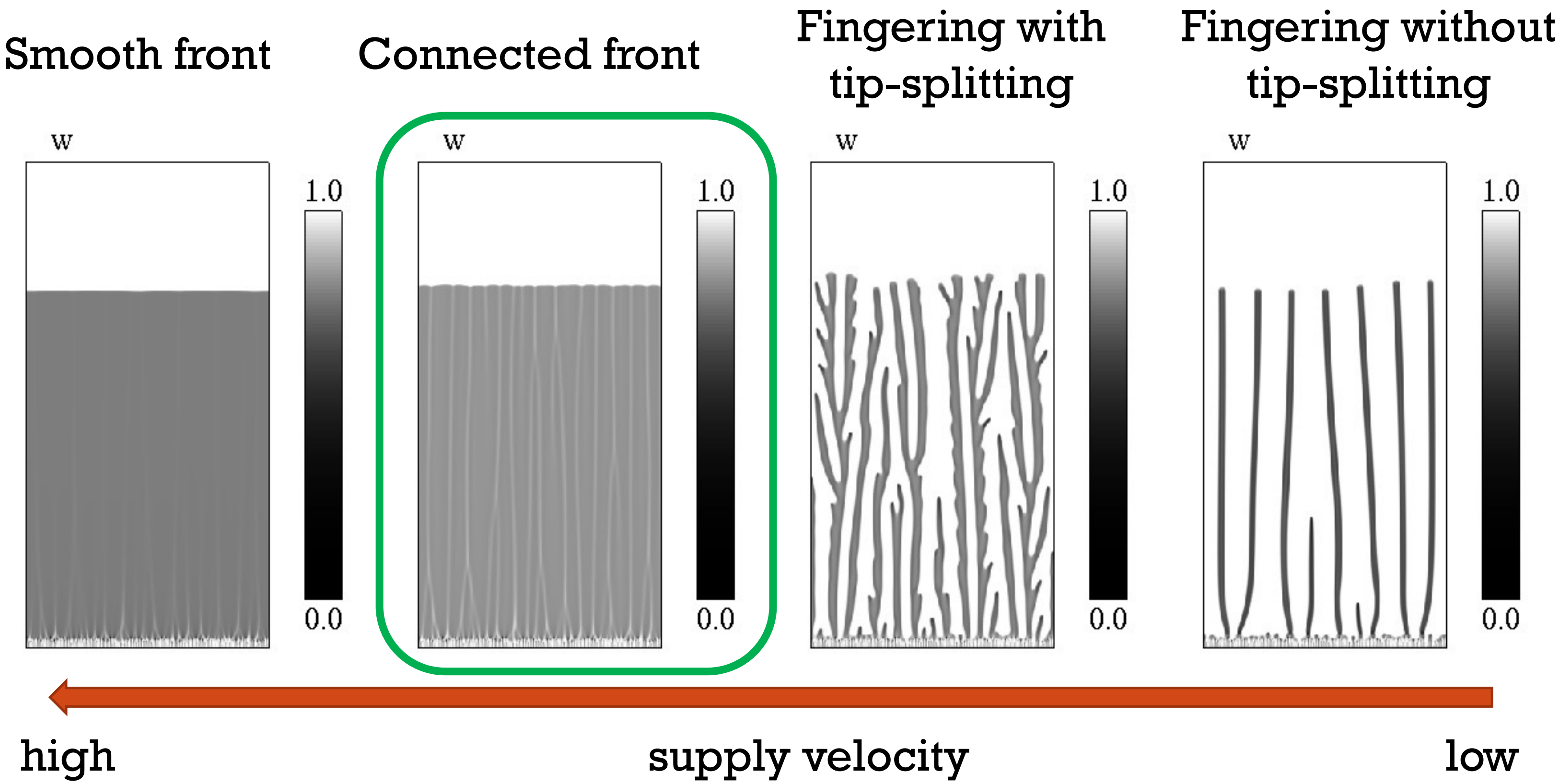
Pe=0.3



Pe=0.17

QUESTIONS

Why do such complicated behaviors appear?

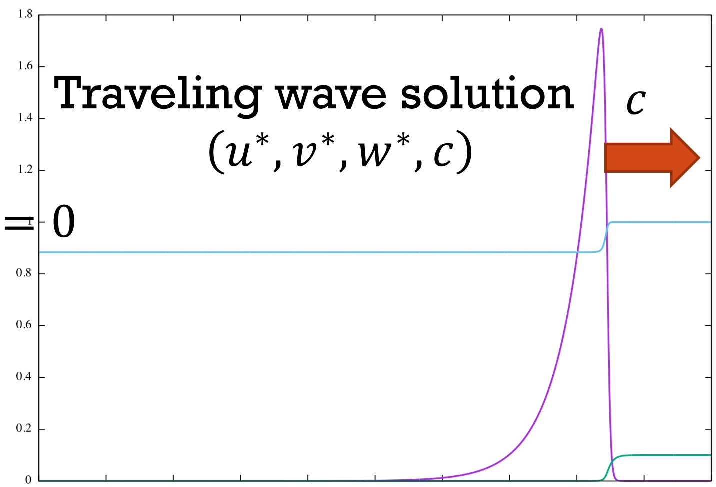


Question:
Can we capture the connected front as an instability of the planar wave?

STABILITY ANALYSIS OF PLANAR WAVE

Introduce a moving coordinate $z = x - ct$:

$$Le(u_{zz} + u_{yy}) + cu_z + \beta\gamma k(u)vw - a(u - \tilde{u}) = 0$$
$$v_{zz} + v_{yy} + (c\phi + \phi Pe)\, v_z - \gamma k(u)vw = 0$$
$$cw_z - H_w\gamma k(u)vw = 0$$



Linearize the system around (u^*, v^*, w^*):

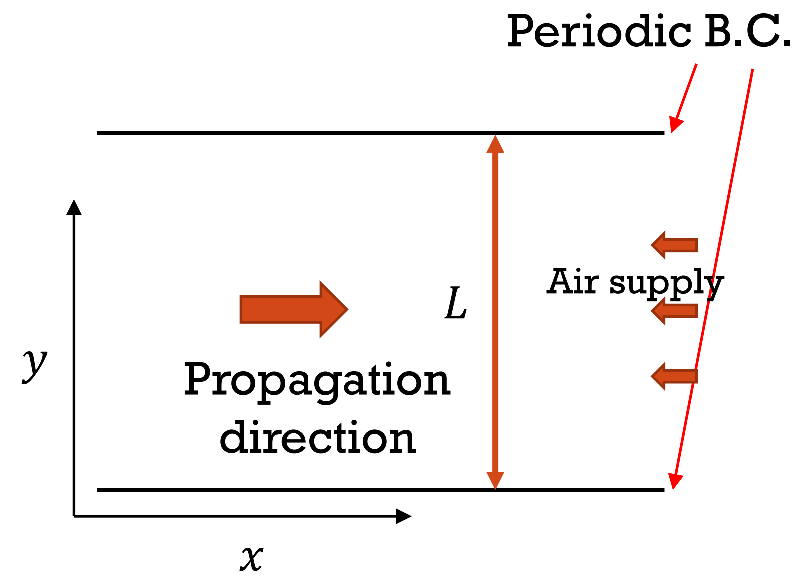
$$\mathcal{L} := \begin{pmatrix} Le\Delta + c\frac{\partial}{\partial z} + \beta\gamma k'(u^*)v^*w^* - a & \beta\gamma k(u^*)w^* & \beta\gamma k(u^*)v^* \\ -\gamma k'(u^*)v^*w^* & \Delta + (c\phi + \phi Pe)\frac{\partial}{\partial z} - \gamma k(u^*)w^* & -\gamma k(u^*)v^* \\ -H_w\gamma k'(u^*)v^*w^* & -H_w\gamma k(u^*)w^* & c\frac{\partial}{\partial z} - H_w\gamma k(u^*)v^* \end{pmatrix}$$

Solve the eigenvalue problem $\mathcal{L}\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \lambda \begin{pmatrix} U \\ V \\ W \end{pmatrix}$.

Eigenfunction of $\mathcal{L}\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \lambda \begin{pmatrix} U \\ V \\ W \end{pmatrix}$ is

$$\begin{pmatrix} U(z,y) \\ V(z,y) \\ W(z,y) \end{pmatrix} = e^{i\frac{2n\pi}{L}y} \begin{pmatrix} \tilde{U}(z) \\ \tilde{V}(z) \\ \tilde{W}(z) \end{pmatrix} \quad (n = 1,2,3\cdots).$$

(Taniguchi & Nishiura`94,
Ikeda, Nagayama & Ikeda. `04)



Therefore,

solve the eigenvalue problem $\tilde{\mathcal{L}}\begin{pmatrix} \tilde{U} \\ \tilde{V} \\ \tilde{W} \end{pmatrix} = \lambda \begin{pmatrix} \tilde{U} \\ \tilde{V} \\ \tilde{W} \end{pmatrix}$,

where

$$\tilde{\mathcal{L}} = \begin{pmatrix} Le\left\{\frac{d^2}{dz^2} - \left(\frac{2n\pi}{L}\right)^2\right\} + c\frac{d}{dz} + \beta\gamma k'(u^*)v^*w^* - a & \beta\gamma k(u^*)w^* & \beta\gamma k(u^*)v^* \\ -\gamma k'(u^*)v^*w^* & \frac{d^2}{dz^2} - \left(\frac{2n\pi}{L}\right)^2 + (c\phi + \phi Pe)\frac{d}{dz} - \gamma k(u^*)w^* & -\gamma k(u^*)v^* \\ -H_w\gamma k'(u^*)v^*w^* & -H_w\gamma k(u^*)w^* & c\frac{d}{dz} - H_w\gamma k(u^*)v^* \end{pmatrix}$$

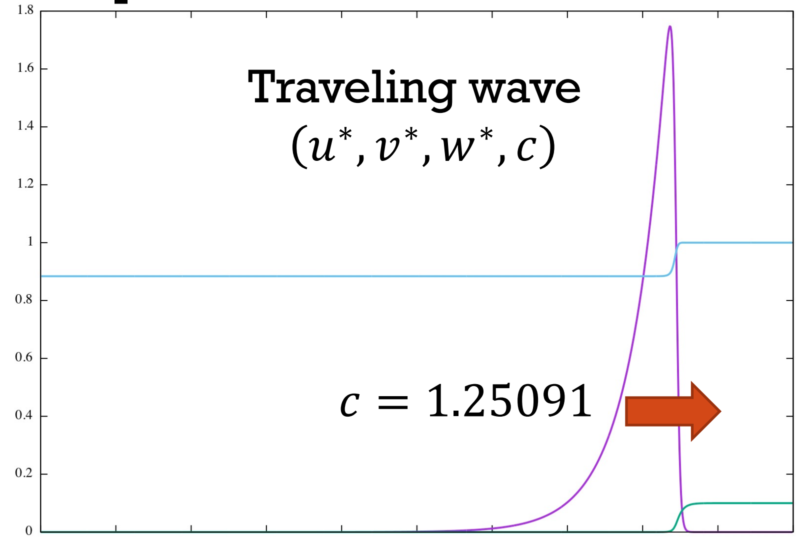
COMPUTATION

$$u_t = Le\Delta u + \beta\gamma k(u)vw - au$$
$$v_t = \Delta v + Pev_x - \gamma k(u)vw$$
$$w_t = -H_w\gamma k(u)vw$$

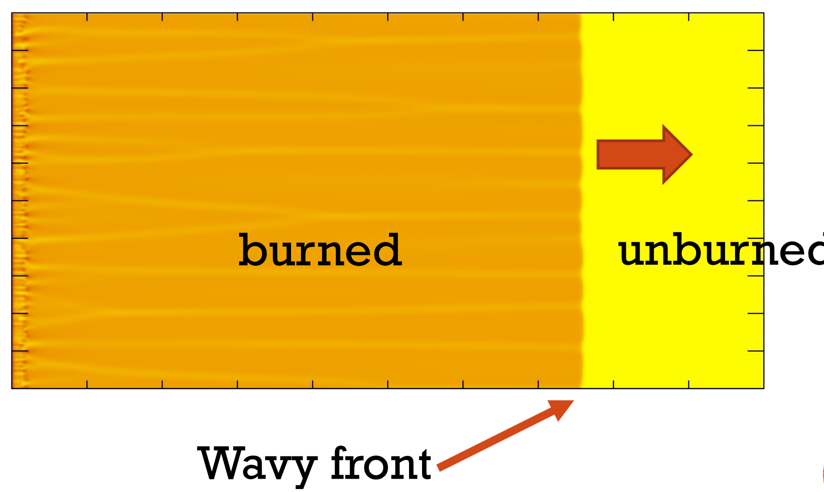
Parameter values :

$$Le = 0.3$$
$$\beta = 20.0$$
$$\gamma = 5.0$$
$$H_w = 1.0$$
$$a = 0.28$$
$$Pe = 0.2$$

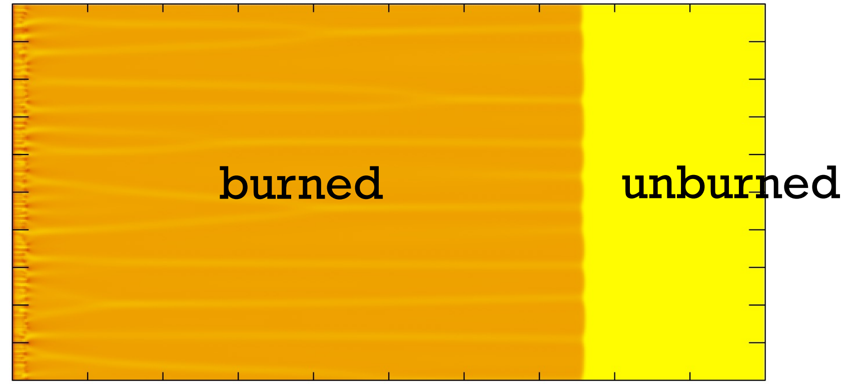
1D problem

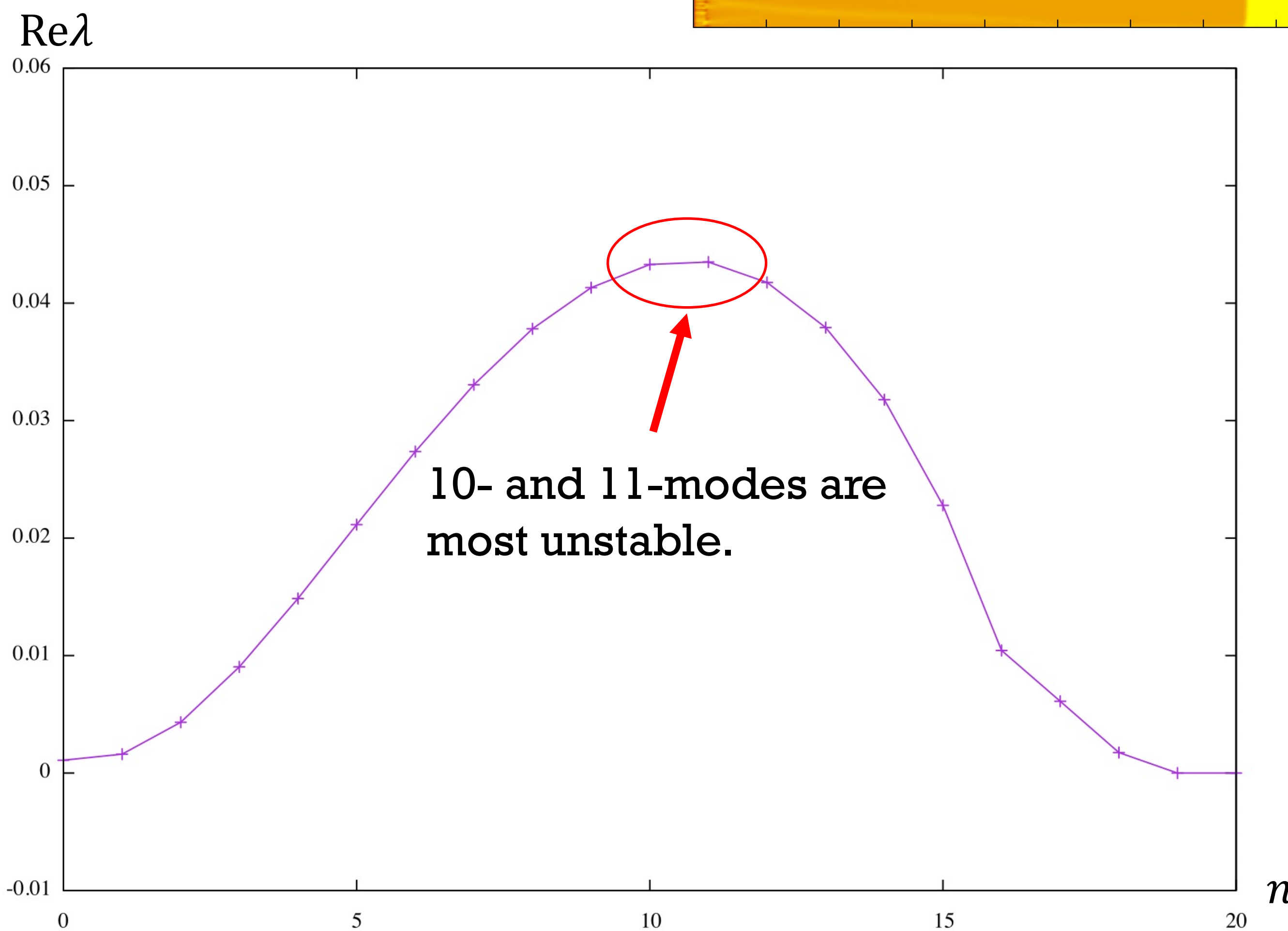


2D problem (w)



EIGENVALUES





Thank you for your kind attention!

"Two phase Stefan problems as the singular limit of competition-diffusion systems arising in population dynamics"

Danielle Hilhorst (Université Paris-Saclay, France)

Competition-diffusion systems are coupled systems of nonlinear parabolic equations, where the unknown functions represent the densities of interacting biological populations. We will first study the singular limit of of a two-component competition-diffusion system in population dynamics when the interspecific competition rate tends to infinity [7], [8]. Using energy estimates, we will prove that the solution converges to the weak solution of a problem with a free boundary, which Mayan Mimura used to call a Stefan problem with zero latent heat [1], [2], [3]. In biological terms, this amounts to proving that the habitats of two interacting populations become completely disjoint in the fast reaction limit. We will then consider a three component competition-diffusion system and prove that its solution converges to a Stefan problem with positive latent heat [4], [6].

Another question involves the limit of the Stefan problem as the latent heat coefficient tends to zero; we will show that it converges to the Stefan problem with zero latent heat [5]. A question which we have started to study is then the following : can we prove a similar result in the case that the partial differential equations in the Stefan problems are perturbed by a white noise in time [9]

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Competition-diffusion sytems and two phase Stefan problems in population dynamics

Danielle Hilhorst

CNRS and Université Paris-Saclay

October 31st 2023

Social life with Mayan

In the occidental world, we always called Mimura sensei Mayan. I met him very long ago, in the Netherlands. Most probably, we first met at a Conference entitled : Conference on Analytical and Numerical approaches to Asymptotic Problems, which took place at the University of Nijmegen, in the Netherlands, on June 9-13, 1980. Since Mayan was an important Professor while I was just a fourth year PhD student, I did not have much chance to discuss with him. But then, on December 2nd 1981, when I defended my doctoral thesis at the University of Leiden, also in the Netherlands, Mayan was there visiting my PhD supervisor Bert Peletier. And Bert requested me to invite Mayan both at my doctoral defense and at the dinner which I was offering at this occasion; it was then a tradition in the Netherlands to organize a PhD dinner on the evening of the defense. This is how my contacts with Mayan started.

Mayan

Social life with Mayan

During many years, I often met Mayan, not only in France and in Japan but also at various conferences all over the world. Mayan was always full of life and happiness and had an incredible creativity. We discussed about mathematics, and in particular about new problems modeling biological phenomena posed by Mayan. I have worked a lot about some of these problems together with PhD students, post-docs as well as well-known mathematicians. I will show you some examples in a little while.

Mayan loved to spend time in France, both for mathematics and everyday life. He also showed an extraordinary hospitality to his colleagues from all over the world that he invited in Japan. We all keep a wonderful memory of these visits to Japan.

The model problem : Dancer, DH, Mimura, Peletier

We consider the competition-diffusion system

$$(\mathcal{P}^k)\begin{cases} \partial_t u = d_1 \Delta u + f(u) - kuv, & \text{in } \Omega \times \mathbb{R}^+ \\ \partial_t v = d_2 \Delta v + g(v) - \alpha kuv, & \text{in } \Omega \times \mathbb{R}^+ \\ \partial_\nu u = 0, \quad \partial_\nu v = 0, & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(\cdot, 0) = u_0^k, \quad v(\cdot, 0) = v_0^k, & \text{on } \Omega, \end{cases}$$

where
$f(s) = \lambda s(1-s), g(s) = \mu s(1-s)$;
$k, \alpha, d_1, d_2, \lambda, \mu$ are positive constants;
$u_0^k, v_0^k \in C(\overline{\Omega}), 0 \leq u_0^k, v_0^k \leq 1$;
$u_0^k \rightharpoonup u_0,\ v_0^k \rightharpoonup v_0$ in $L^2(\Omega)$ as $k \to \infty$.

u, v are the densities of two biological populations;
λ, μ are the intraspecific competition rates;
$k, \alpha k$ are the interspecific competition rates.

The main question

Our main question:

Let (u^k, v^k) denote the solution of Problem $(\mathcal{P}^k)$. What is the singular limit of the solution pair (u^k, v^k) as $k \to \infty$?

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A priori bounds

By a solution of Problem $(\mathcal{P}^k)$ we mean a pair of functions (u^k, v^k) such that $u^k, v^k \in C(\overline{Q}) \cap C^{2,1}(\overline{\Omega} \times [\delta, T])$ for all $\delta \in (0, T)$. We begin with a priori bounds for solutions of Problem $(\mathcal{P}^k)$.

Lemma. Let (u^k, v^k) be a solution of Problem $(\mathcal{P}^k)$. Then $0 \leq u^k \leq 1$ and $0 \leq v^k \leq 1$ in $\overline{Q}$, where $Q := \Omega \times \mathbb{R}^+$.

Proof. This assertion follows from the maximum principle.

Existence and uniqueness - a priori estimates

The existence and uniqueness of the solution (u^k, v^k) of Problem $(\mathcal{P}^k)$ follows from [Lunardi, Proposition 7.3.2].

Next we obtain a priori bounds for the solution (u^k, v^k) of Problem $(\mathcal{P}^k)$ which are *uniform* with respect to the parameter k in the equations. This will enable us to study the properties of the family of solutions (u^k, v^k) for large values of k.

Lemma. We have

$$\int_0^T \int_\Omega u^k v^k \leq \frac{|\Omega|}{k}(\ell_0 T + 1).$$

Proof. Integration of the equation for u^k over $Q_T := \Omega \times (0, T)$ yields

$k \int_0^T \int_\Omega u^k v^k = d_1 \int_0^T \int_{\partial\Omega} \frac{\partial u^k}{\partial \nu} + \int_0^T \int_\Omega f(u^k) u^k - \int_\Omega u^k(T) + \int_\Omega u_0^k$

$\leq (\ell_0 T + 1)|\Omega|.$

A priori estimates

Lemma. There exists a positive constant $\mathcal{C}$, which does not depend on k, such that

$$\int_0^T \int_\Omega |\nabla u^k|^2 \le \mathcal{C},$$

and

$$\int_0^T \int_\Omega |\nabla v^k|^2 \le \mathcal{C}.$$

Proof. We multiply the first equation in $(\mathcal{P}^k)$ by u and integrate on Ω. This yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_k^2 + d_1 \int_\Omega |\nabla u_k|^2 + k \int_\Omega u_k^2 v_k \le \ell_0 |\Omega|,$$

where we have used the bounds on u^k and v^k. When we integrate on $(0, T)$ we obtain the first estimate. The second estimate can be proved in a similar way.

A way to eliminate the large parameter k

Next we consider the function

$$z^k = u^k - \frac{1}{\alpha} v^k,$$

which appears when we want to eliminate the terms involving k from the partial differential equation. It satisfies

$$z_t^k = d_1 \Delta u^k - \frac{d_2}{\alpha} \Delta v^k + u^k f(u^k) - \frac{1}{\alpha} v^k g(v^k) \text{ in } Q_T$$

together with the homogeneous Neumann boundary condition

$$\frac{\partial z^k}{\partial \nu} = 0 \text{ on } S_T := \partial\Omega \times (0, T).$$

Where do we stand now?

We have $L^\infty(Q_T)$ estimates for u^k and v^k, and $L^2(Q_T)$ estimates for $|\nabla u^k|$ and $|\nabla v^k|$.

This is not sufficient to pass to the limit as $k \to \infty$, even though it would be if we would work on an elliptic problem. Here we need some extra knowledge about either the time derivative of the solution pair or differences of time translates of the solution pair.

We apply the Fréchet-Kolmogorov Theorem (see for instance the book of Brezis on functional analysis).

Limit as $k \to \infty$

We deduce that the sequences $\{u_k\}$ and $\{v_k\}$ are relatively compact in $L^2(Q_T)$. In particular, there exist subsequences of $\{u^k\}$ and $\{v^k\}$, which we denote again by $\{u^k\}$ and $\{v^k\}$, and functions $\overline{u}$ and $\overline{v}$ in $L^2(0, T; H^1(\Omega))$ such that, as $k \to \infty$, u^k and v^k converge to their limits $\overline{u}$ and $\overline{v}$ strongly in $L^2(Q_T)$, a. e. in Q_T and weakly in $L^2(0, T; H^1(\Omega))$.

Characterization of the limit functions

Lemma. Let T be an arbitrary positive number. The limit functions $(\overline{u}, \overline{v})$ are such that

$$\int_0^T \int_\Omega \Big\{ \Big(\overline{u} - \frac{1}{\alpha}\overline{v} \Big) \varphi_t - \nabla \Big(d_1 \overline{u} - \frac{d_2}{\alpha}\overline{v} \Big) \nabla\varphi + \Big(\overline{u} f(\overline{u}) - \frac{1}{\alpha}\overline{v} g(\overline{v}) \Big) \varphi \Big\}$$

$$= - \int_\Omega (u_0 - \frac{v_0}{\alpha}) \varphi(0),$$

for all functions $\varphi \in C^\infty(Q_T)$ such that $\varphi(T) = 0$.

Proof. When we multiply the equation for z^k by a test function $\varphi \in C_0^\infty(Q_T)$ such that $\varphi(T) = 0$, and integrate by parts, we obtain the identity

$$\int_0^T \int_\Omega \Big\{ \Big(u^k - \frac{1}{\alpha} v^k \Big) \varphi_t - \nabla \Big(d_1 u^k - \frac{d_2}{\alpha} v^k \Big) \nabla\varphi$$

$$+ \Big(u^k f(u^k) - \frac{1}{\alpha} v^k g(v^k) \Big) \varphi \Big\} = - \int_\Omega (u_0^k - \frac{v_0^k}{\alpha}) \varphi(0).$$

Characterization of the limit functions

We now let $k \to \infty$ along a sequence for which u^k and v^k strongly converge in $L^2(Q_T)$ and a. e. to their limits. Then, because

$$u^k \to \overline{u}, \quad \text{and} \quad v^k \to \overline{v} \quad \text{as} \;\; k \to \infty \quad \text{a.e.} \;\; \text{in} \;\; Q_T,$$

and $|u^k|, |v^k| \leq 1$ for all $k \geq 1$, it follows by the dominated convergence theorem that

$$\int_0^T \int_\Omega u^k f(u^k) \to \int_0^T \int_\Omega \overline{u} f(\overline{u}) \quad \text{as} \quad k \to \infty.$$

A similar result holds for the sequence $\{v^k g(v^k)\}$. Passing to the limit in the equality above permits to complete the proof.

Characterization of the limit functions

Next we show that the function

$$z := \overline{u} - \frac{1}{\alpha}\overline{v},$$

is a weak solution of Problem $(\mathcal{P})$ defined by

$$(\mathcal{P}) \begin{cases} \partial_t z = \nabla(d(z)\nabla z) + h(z), & \textit{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu z = 0, & \textit{on } \partial\Omega \times \mathbb{R}^+, \\ z(\cdot, 0) = z_0 := u_0 - \frac{v_0}{\alpha}, & \textit{on } \Omega, \end{cases}$$

where

$$d(s) = \{d_1 \quad \text{if} \quad s > 0, \quad d_2 \quad \text{if} \quad s < 0\},$$

and

$$h(s) = \{f(s)s \quad \text{if} \quad s > 0, \quad g(-\alpha s)s \quad \text{if} \quad s < 0\}.$$

Characterization of the limit functions

Definition. A function z is called a weak solution of Problem $(\mathcal{P})$ if

$$z \in L^\infty(\Omega \times \mathbb{R}^+) \cap L^2(0, T, H^1(\Omega));$$

$$-\int_0^T \int_\Omega \{z\varphi_t - d(z)\nabla z \nabla\varphi + h(z)\varphi\} = \int_\Omega z_0\varphi(0),$$

for all functions $\varphi \in C^\infty(Q_T)$ such that $\varphi(T) = 0$ and for all $T > 0$.

Characterization of the limit functions

Lemma. The function z is a weak solution of Problem $(\mathcal{P})$.

Proof. We already know that $z \in L^\infty(\Omega \times \mathbb{R}^+)$ and that $z \in L^2(0, T; H^1(\Omega))$.

We observe that

$$d_1 \, \nabla \overline{u} - \frac{d_2}{\alpha} \nabla \overline{v} = d(z) \nabla z$$

and that

$$\overline{u} f(\overline{u}) - \frac{\overline{v}}{\alpha} g(\overline{v}) = h(z).$$

Therefore z satisfies the integral equality

$$\int_0^T \int_\Omega \{ z \varphi_t - d(z) \nabla z \nabla \varphi + h(z) \varphi \} = \int_\Omega (u_0 - \frac{v_0}{\alpha}) \varphi(0)$$

for all functions $\varphi \in C^\infty(Q_T)$ such that $\varphi(T) = 0$ and for all $T > 0$.

Characterization of the limit functions

Thus z is a weak solution of Problem $(\mathcal{P})$.

We also have the following result.

Lemma. Problem $(\mathcal{P})$ has exactly one weak solution z, and $z \in C^{\beta,\beta/2}(\overline{\Omega} \times [0,\infty))$ for all $\beta \in (0,1)$.

The limit problem as a free boundary problem

The limit problem is a free boundary problem. The free boundary separates the regions where $\{u > 0, v = 0\}$ and $\{v > 0, u = 0\}$.

The limit free boundary problem

We can give both

- a weak form, where the free boundary does not explicitely appear;
- a strong form, with explicit boundary conditions.

The convergence result

We have proved that as $k \to \infty$,

$$u^k \to z^+, \quad v^k \to \alpha z^-,$$

strongly in $L^2(Q_T)$, where the function z is the unique weak solution of the problem

$$(\mathcal{P}) \begin{cases} \partial_t z = \Delta \mathcal{D}(z) + h(z), & \textit{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu z = 0, & \textit{on } \partial\Omega \times \mathbb{R}^+, \\ w(\cdot, 0) = z_0 := u_0 - \dfrac{v_0}{\alpha}, & \textit{on } \Omega, \end{cases}$$

with $\mathcal{D}(s) := d_1 s^+ - d_2 s^-$ and $h(s) := f(s^+) - g(\alpha s^-)$, where $s^+ = max(s, 0)$ and $s^- = -min(s, 0)$.

The strong form of the limit free boundary problem

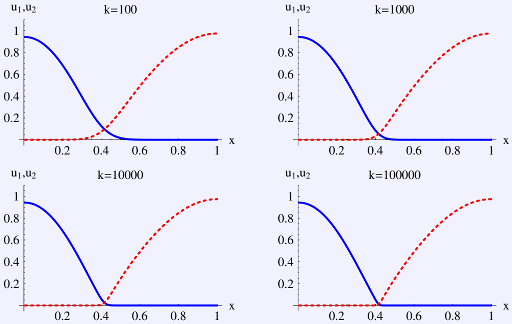
Assume that, at each time $t \in [0, T]$, there exists a close hypersurface $\Gamma(t)$ and two subdomains $\Omega_u(t)$, $\Omega_v(t)$ such that

$$\overline{\Omega} = \overline{\Omega_u(t)} \cup \overline{\Omega_v(t)}, \quad \Gamma(t) = \overline{\Omega_u(t)} \cap \overline{\Omega_v(t)},$$
$$z(\cdot, t) > 0 \quad \text{on } \Omega_u(t), \qquad z(\cdot, t) < 0 \quad \text{on } \Omega_v(t).$$

Assume furthermore that $t \mapsto \Gamma(t)$ is smooth enough and that $(u, v) := (z^+, \alpha z^-)$ are smooth up to $\Gamma(t)$. Then the functions u and v satisfy

$$(\mathcal{P}) \begin{cases} \partial_t u = d_1 \Delta u + f(u) & \text{in } \; Q_u := \bigcup \{\Omega_u(t), t \in [0, T]\} \\ \partial_t v = d_2 \Delta v + g(v) & \text{in } \; Q_v := \bigcup \{\Omega_v(t), t \in [0, T]\} \\ u = v = 0 & \text{on } \Gamma := \bigcup \{\Gamma(t), t \in t \in [0, T]\} \\ d_1 \partial_n u = -\dfrac{d_2}{\alpha} \partial_n v & \text{on } \; \Gamma \\ \partial_\nu v = 0 & \text{on } \; \partial\Omega \times [0, T] \\ + \; \text{initial conditions.} & \end{cases}$$

The two-component system



Comments on the limit free boundary problem

This is a Stefan problem with zero latent heat.

This led us to search for a reaction-diffusion system whose solution converges to that of a "real" Stefan problem

D. Hilhorst, M. Iida, M. Mimura, H. Ninomiya, *Japan J. Ind. Appl. Math.* (2001)

To that purpose we need

- A three component system,
- With two partial differential equations coupled to an ordinary differential equation.

A three-component reaction-diffusion system

We consider the system

$$
(\mathcal{Q}^k)\begin{cases}
\partial_t u = d_1 \Delta u + f(u) - k s_1 uv - k\lambda s_1 (1-w)u & \text{in } \Omega \times \mathbb{R}^+ \\
\partial_t v = d_2 \Delta v + g(v) - k s_2 uv - k\lambda s_2 wv & \text{in } \Omega \times \mathbb{R}^+ \\
\partial_t w = k(1-w)u - kwv & \text{in } \Omega \times \mathbb{R}^+ \\
\partial_\nu u = 0, \quad \partial_\nu v = 0, & \text{on } \partial\Omega \times \mathbb{R}^+ \\
u(\cdot,0) = u_0^k, \quad v(\cdot,0) = v_0^k, \quad w(\cdot,0) = w_0^k & \text{on } \Omega,
\end{cases}
$$

$u_0^k, v_0^k \in C(\overline{\Omega}),\ w_0^k \in L^\infty(\Omega)$,
$0 \le u_0^k, v_0^k, w_0^k \le 1$;
$u_0^k \rightharpoonup u_0,\ v_0^k \rightharpoonup v_0,\ w_0^k \rightharpoonup w_0$ in $L^2(\Omega)$ as $k \to \infty$.
The function w^k approximates the characteristic function of the habitat of the population u^k.

The weak form of the limit free boundary problem

$$(\mathcal{Q}^\lambda)\begin{cases} \partial_t z = \Delta \mathcal{D}(\varphi_\lambda(z)) + h(\varphi_\lambda(z)) \text{ in } \Omega \times \mathbb{R}^+ \\ \partial_\nu \mathcal{D}(\varphi_\lambda(z)) = 0 \text{ on } \partial\Omega \times \mathbb{R}^+ \\ z(\cdot, 0) = \dfrac{u_0}{s_1} - \dfrac{v_0}{s_2} + \lambda w_0 \text{ on } \Omega, \end{cases}$$

with

$\mathcal{D}(r) := d_1 r^+ - d_2 r^-$, $\quad h(r) := \dfrac{f(s_1 r^+)}{s_1} - \dfrac{g(s_2 r^-)}{s_2}$, $\quad \varphi_\lambda(r) = (r - \lambda)^+ - r^-$.

We now have a Stefan problem with positive latent heat λ. We also define the limit functions

$$u = s_1 \varphi_\lambda(z^+), \quad v = s_2 \varphi_\lambda(z^-), \quad w = \frac{z - \varphi_\lambda(z)}{\lambda}.$$

The strong form of the Stefan problem with positive latent heat

The functions u and v satisfy

$$(\mathcal{Q}^\lambda)\begin{cases} \partial_t u = d_1 \Delta u + f(u) & \text{in } \ Q_u := \bigcup \{\Omega_u(t), t \in [0,T]\} \\ \partial_t v = d_2 \Delta v + g(v) & \text{in } \ Q_v := \bigcup \{\Omega_v(t), t \in [0,T]\} \\ u = v = 0 & \text{on } \Gamma := \bigcup \{\Gamma(t), t \in [0,T]\} \\ \lambda V_n = -\dfrac{d_1}{s_1}\partial_n u - \dfrac{d_2}{s_2}\partial_n v & \text{on } \ \Gamma \\ \partial_\nu v = 0 & \text{on } \ \partial\Omega \times [0,T] \\ + \ \text{initial conditions.} \end{cases}$$

Present work with Ciotir, El Kettani and Goreac

With Mayan Mimura and Reiner Schätzle, we have proved that, as $\lambda \to 0$, the weak solution z^λ of Problem $(\mathcal{Q}^\lambda)$ converges to the weak solution z of Problem $(\mathcal{P})$.

Hilhorst, Danielle; Mimura, Masayasu; Schätzle, Reiner, Vanishing latent heat limit in a Stefan-like problem arising in biology, Nonlinear Anal. Real World Appl. 4 (2003), no. 2, 261–285.

With Ciotir, El Kettani and Goreac, we are adding a multiplicative noise involving a white noise in time on the right-hand-side of the partial differential equation for z^λ, and try to prove a similar result. This study is rather technical but I believe that it will work out.

Mayan, may you rest in peace.

"The floodgates to pattern formation problems"

Yasumasa Nishiura (Hokkaido University, Japan)

W. T. Gowers, a Fields medalist from Cambridge in 1998, eloquently expounded on the pivotal role of Paul Erdős in his essay titled "The Two Cultures of Mathematics." He (Erdős) is famous not because it has large numbers of applications, nor because it is difficult, nor because it solved a long-standing open problem. Its fame rests on the fact that it opened the floodgates to probabilistic arguments in combinatorics.

In a similar vein, Mayan Mimura opened a parallel set of floodgates, ones that lead to the modeling and analysis of reaction-diffusion equations. I am eager to trace the initial footsteps of Mayan and delve into the lasting impact of his contributions on present-day research.

HOKKAIDO UNIVERSITY

TOHOKU UNIVERSITY

The floodgates to pattern formation problems

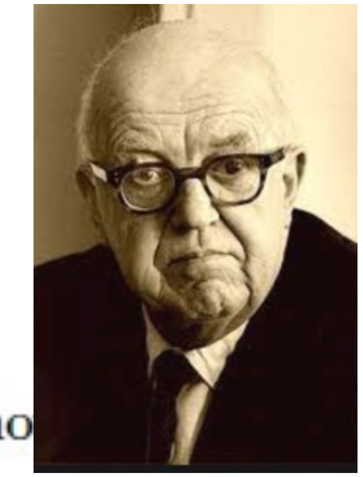
Yasumasa Nishiura

ICMMA 2023 at Meiji University
“In memory of Professor Mayan Mimura”

Encounter

- Encounter with Mayan in early 70's (Konan Univ)
  - Prof. Yamaguti's seminar
  - Debate on "what is a good math model?"
- Scientific collaboration in 80's and early 90's (Hiroshima Univ) and discussion about the embryonic stage of the issues:
  - On the necessity to establish an applied mathematics major.
  - Importance of computational approach
  - How should mathematics establish its relationship with various sciences? （諸科学との関係）
  - Combining science and the humanities is quite challenging. （文理融合は簡単ではない）

C.P. Snow "The Two Cultures and the Scientific Revolution"



「２つの文化と科学革命」

人文的文化　vs　科学的文化

(The arts vs Sciences)

In his famous Rede lecture of 1959, entitled "The Two Cultures", C. P. Sno that the lack of communication between the humanities and the sciences was very harmful, and he particularly criticized those working in the humanities for their lack of understanding of science. One of the most memorable passages draws attention to a lack of symmetry which still exists, in a milder form, forty years later:

> A good many times I have been present at gatherings of people who, by the standards of the traditional culture, are thought highly educated and who have with considerable gusto been expressing their incredulity at the illiteracy of scientists. Once or twice I have been provoked and have asked the company how many of them could describe the Second Law of Thermodynamics. The response was cold: it was also negative. Yet I was asking something which is about the scientific equivalent of: *Have you read a work of Shakespeare's?*

"Third culture" by John Brockman

Thomas Kuhn "Paradigm Shift"

- Those who a ... paradigm
through lon ... es, engage
in scientific ... rm of
puzzle-solvi ... adigm.
Much of sci ... nce, is
nothing but
- However, paradigms are not always secure indefinitely. Puzzles gradually run out, and, conversely, anomalies that cannot be dealt with by the paradigm accumulate. This eventually leads to a crisis of the paradigm, and from the midst of that confusion, a new paradigm emerges, leading to a "scientific revolution."
- It is not clear what is a paradigm in mathematics, namely once it was proved rigorously, it becomes eternal. How about Kurt Gödel?

Thomas, K. (1962). *The structure of scientific revolutions* Chicago: University of Chicago Press.

The Two Cultures of Mathematics.

W. T. Gowers

Problem solver vs Theory builder

W. T. Gowers is a British mathematician (combinatorialist) at Cambridge awarded Fields medal 1998. Since 2020, he is a professor at the Collège de France, born in 1963.

Criticisms of Combinatorics

(from core mathematics)

1. Lacks direction, or goals of a general kind
2. Not particularly deep
3. No interesting connections to other parts of mathematics (core mathematics)
4. Many of them do not have applications

Moreover, mathematicians in the theory-building areas often regard what they are doing as the central core of mathematics, with subjects such as combinatorics thought of as peripheral and not particularly relevant to the main aims of mathematics.

One can almost imagine a gathering of highly educated mathematicians expressing their incredulity at the ignorance of combinatorialists, most of whom could say nothing intelligent about quantum groups, mirror symmetry, Calabi-Yau manifolds, the Yang-Mills equation, solitons or even cohomology.

流れるままに

These criticisms can be answered in a similar way. Consider first the notion that there are not general goals in combinatorics. I quote again from the interview with Atiyah [A1]:
"I was thinking more of the tendency today for people to develop whole areas of mathematics on their own, in a rather abstract fashion. They just go on beavering away. *（せっせと働く）*
If you ask what is it all for, what is its significance, what does it connect with you and that they don't know.

Atiyah was not particularly referring to combinatorics, but he makes a powerful point, and it is as important for combinatorialists as it is for anyone else to show that they are doing more than merely beavering away.

An Interview With Michael Atiyah

Michael Atiyah was born in 1929 and received his B.A. and Ph.D. from Trinity College, Cambridge (1952, 1955). During his career he has been Savilian Professor of Geometry at Oxford (1963–69) and Professor of Mathematics at the Institute for Advanced Study in Princeton (1969–72); he is currently a Royal Society Research Professor of Mathematics at Oxford University.

Among other honors Professor Atiyah is a Fellow of the Royal Society and a member of the National Academies of France, Sweden, and the USA. He received the Fields Medal at the International Congress of Mathematicians held in Moscow, 1966. His research interests span a broad area of mathematics including topology, geometry, differential equations and mathematical physics.

The following is an edited version of an interview in Oxford with Roberto Minio, former editor of The Intelligencer.

Why should problem-solving subjects be less highly regarded than theoretical ones? To answer this question we must consider a more fundamental one: what makes one piece of mathematics more interesting than another? Once again, Atiyah writes very clearly and sensibly on this matter (while acknowledging his debt to earlier great mathematicians such as Poincar e and Weyl). He makes the point (see for example [A2]) that so much mathematics is produced that it is not possible for all of it to be remembered. The processes of abstraction and generalization are therefore very important as a means of making sense of the huge mass of raw data (that is, proofs of individual theorems) and enabling at least some of it to be passed on. The results that will last are the ones that can be organized coherently and explained economically to future generations of mathematicians. Of course, some results will be remembered because they solve very famous problems, but even these, if they do not into an organizing framework, are unlikely to be studied in detail by more than a handful of mathematicians.

抽象化と一般性がなければ，次へは進めない．

The important ideas of combinatorics do not usually appear in the form of precisely stated theorems, but more often as general principles of wide applicability.

Combinatorics ではその現れ方は違う．例えばErdos の例では

This result of Erdos [E] is famous not because it has large numbers of applications, nor because it is difficult, nor because it solved a long-standing open problem. Its fame rests on the fact that it opened the floodgates to probabilistic arguments in combinatorics.

Theorem. *For every positive integer k there is a positive integer N, such that if the edges of the complete graph on N vertices are all coloured either red or blue, then there must be k vertices such that all edges joining them have the same colour.*

SOME REMARKS ON THE THEORY OF GRAPHS

P. ERDÖS

The present note consists of some remarks on graphs. A graph G is a set of points some of which are connected by edges. We assume here that no two points are connected by more than one edge. The complementary graph G' of G has the same vertices as G and two points are connected in G' if and only if they are not connected in G.

A special case of a theorem of Ramsey can be stated in graph theoretic language as follows:

There exists a function $f(k, l)$ of positive integers k, l with the following property. Let there be given a graph G of $n \geqq f(k, l)$ vertices. Then either G contains a complete graph of order k, or G' a complete graph of order l. (A complete graph is a graph any two vertices of which are connected. The order of a complete graph is the number of its vertices.)

It would be desirable to have a formula for $f(k, l)$. This at present we can not do. We have however the following estimates:

THEOREM I. *Let $k \geqq 3$. Then*

$$2^{k/2} < f(k, k) \leqq C_{2k-2,k-1} < 4^{k-1}.$$

The least integer N that works is known as $R(k)$.

Lower bound: $R(k) \geq 2^{k/2}$ **probabilistic method**

ゴールが明確に示されている分野もある

Some branches of mathematics are dominated by a small number of problems of universally acknowledged importance. One can justify many results by saying that, in however small a way, they shed light on the Riemann hypothesis, the Birch-Swinnerton-Dyer conjecture, Thurston's geometrization conjecture, the Novikov conjecture or something of the kind.

Combinatorics では少しそれとは異なる

It would be difficult to demonstrate that combinatorics had many general goals of the sort just mentioned (with the one exception of the P=NP problem). However, just as the true significance of a result in combinatorics is very often not the result itself, but something less explicit that one learns from the proof, so the general goals of combinatorics are not always explicitly stated.

Something similar?

The dichotomy of Applied Math vs Pure Math

How do you select a problem to study?

MINIO: How do you select a problem to study?

ATIYAH: I think that presupposes an answer. I don't think that's the way I work at all. Some people may sit back and say, "I want to solve this problem" and they sit down and say, "How do I solve this problem?" I don't. I just move around in the mathematical waters, thinking about things, being curious, interested, talking to people, stirring up ideas; things emerge and I follow them up. Or I see something which connects up with something else I know about, and I try to put them together and things develop. I have practically never started off with any idea of what I'm going to be doing or where it's going to go. I'm interested in mathematics; I talk, I learn, I discuss and then interesting questions simply emerge. I have never started off with a particular goal, except the goal of understanding mathematics.

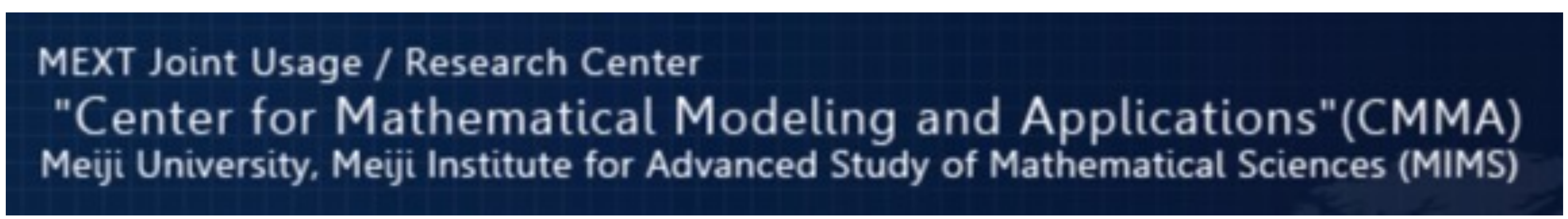
[23] R. Minio: "An interview with Michael Atiyah," *Pokroky Mat. Fyz. Astronom.* **31** : 3 (1986), pp. 154–168. Czech translation of article from *Math. Intell.* **6**:1 (1984). MR 857260

Floodgates

- Finding of new things:
  - Finding of new phenomena
    - Quasi-crystal, chaos, …
  - Finding new (simple) model
    - Landau model, Kuramoto model, …
  - Finding of new methods
- Connecting two different concepts
  - Important in Mathematics
    - Trigger to explore a new world
  - Introduce a new insight to a hard problem (like Paul Erdos)
- Establish a common platform for applied mathematicians
  - Collaborations with interdisciplinary people
  - Incubator in all directions
  - Mayan was skilled at conversation and good at winning people over.

MIMS

- Mayan's dream of Applied Math Dept. with many twists and turns starting around late 80's, tough negotiations with deans and presidents
  - Hiroshima Univ.
    - Dept. of Mathematical and Life Sciences (1999)
      数理分子生命理学専攻
    - Graduate School of Integrated Sciences for Life (2019)
      Program of Mathematical and Life Sciences
  - Meiji Univ.
    - MIMS was established!



    - Combining science and the humanities is quite challenging. （文理融合は簡単ではない）

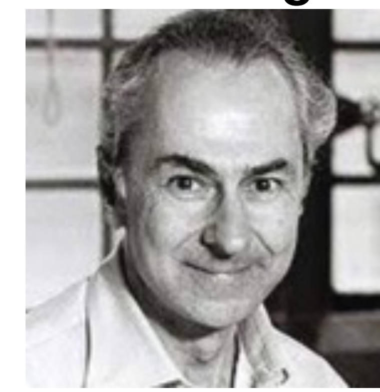
"Ecocultural range-expansion model of modern humans in Paleolithic" by Joe Yuichiro Wakano (Archaeology based on ancient DNA analysis)

Soft science vs Hard science

Soft science vs Hard science

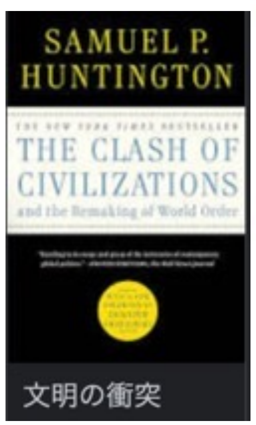
S. Huntington.

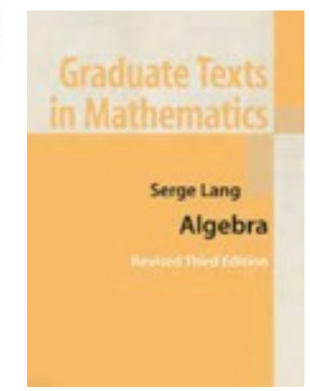
S. Lang



Soft sciences are often harder than hard sciences
Discover (1987, August) by Jared Diamond

相互理解のギャップがもたらす不幸





- "The overall correlation between frustration and instability (in 62 countries of the world) was 0.50." --Samuel Huntington, professor of government, Harvard

- "This is utter nonsense. How does Huntington measure things like social frustration? Does he have a social-frustration meter? I object to the academy's certifying as science what are merely political opinions." -- Serge Lang, professor of mathematics, Yale

"pseudo Mathematics" by Huntington

- "What does it say about Lang's scientific standards that he would base his case on twenty-year-old gossip?" . . . "a bizarre vendetta" . . . "a madman . . ." -- Other scholars, commenting on Lang's attack

Lang vs. Huntington might seem like just another silly blood-letting in the back alleys of academia, hardly worth anyone's attention. But this particular dogfight is an important one. Beneath the name calling, it has to do with a central question in science: Do the so-called soft sciences, like political science and psychology, really constitute science at all, and do they deserve to stand beside "hard sciences," like chemistry and physics?

But NAS is more than an honorary society; it's a conduit for advice to our government. As to the relative importance of soft and hard science for humanity's future, there can be no comparison. It matters little whether we progress with understanding the diophantine approximation. Our survival depends on whether we progress with understanding how people behave, why some societies become frustrated, whether their governments tend to become unstable, and how political leaders make decisions like whether to press a red button. Our National Academy of Sciences will cut itself out of intellectually challenging areas of science, and out of the areas where NAS can provide the most needed scientific advice, if it continues to judge social scientists from a posture of ignorance.

(NAS: National Academy of Sciences)

Summary

- The Two Cultures: C.P. Snow
  - The arts vs Sciences:
- The two cultures of Mathematics: W.T. Gowers
  - Michael Atiya’s Interview
- Soft sciences vs Hard sciences: Jared Diamond
  - Huntington vs Serge Lang
- Border between pure and appl is disappearing.
  - Appl. Math broadens the spectrum of pure math.
  - Appl math demands all mathematics.

There still remains a gap, however,

- Mayan Mimura opened the floodgates to pattern formation problems!
  - New modeling in mathematical biology
  - New collaborations among interdisciplinary fields
  - New institute where two cultures meet

Above all, it has bestowed upon me the joy and delight of scholarly exploration

そして多くの研究の喜びと楽しさを伝えてくれた

1988@Hiroshima

Mayan's cottage

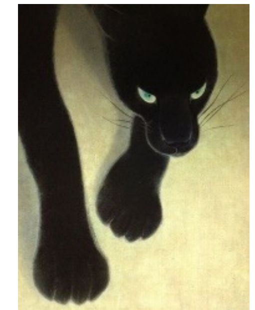
Ogawa-san's cottage
2010, September

寺本英

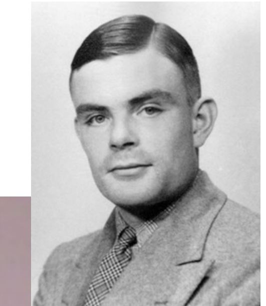
岡田節人

山口昌哉

Mathematical biology
Numerical Analysis
Pattern dynamics
Mathematical modeling



山口華楊



Jim Murray

M. Bertch
V. Cappasso
Daniell Hilhorst

西田孝明
野木達夫
藤井宏
亀高惟倫

蔵本由紀

三村昌泰

Man of action

山口智彦

自己組織化

重定奈々子
厳佐庸
望月敦史

Singular perturbation
Interfacial Dynamics

学位論文

Bifurcation theory
Symmetry-breaking

田端正久
俣野博
小林亮
岡本久
柳田英二

太田隆夫

甲斐昌一

Thank you for listening!

"Keller-Segel type approximation for nonlocal Fokker-Planck equations in one-dimensional bounded domain"

Yoshitaro Tanaka (Future University Hakodate, Japan)

To describe biological phenomena such as cell migration and cell adhesion many evolutional equations are proposed in which an advective convolution term with a suitable integral kernel is imposed. It is well known that such nonlocal equations can reproduce various behaviors depending on the shape of the integral kernel. These nonlocal evolutional equations are often difficult to analyze, and the analytical method is developing. In the light of these background we approximate the nonlocal Fokker-Planck equations by the combination of a Keller-Segel system which is a typical and locally dynamical system. We will show that the solution of the nonlocal Fokker-Planck equation with any even continuous integral kernel can be approximated as a singular limit of the Keller-Segel system with specified parameters.

ICMMA 2023, ″Reaction-diffusion systems: from the past to the future ″ in memory of Prof. Masayasu Mimura, Meiji University, 1 Nov. 2023. Chair: Toshiyuki OGAWA

Keller-Segel type approximation for nonlocal Fokker-Planck equations in one-dimensional bounded domain

Speaker：〇Yoshitaro Tanaka（Future University Hakodate）

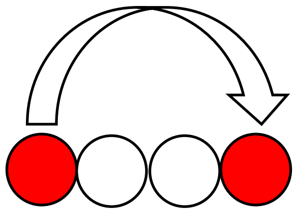
Collaborator： Hideki Murakawa (Ryukoku University)

Dedicated to the memory of Professor Masayasu Mimura

Supported by JSPS, Grant number JP20K1436

Nonlocal interactions

Nonlocal interactions (spatially long range interactions) have attracted attentions in various fields:



◉ Neural firing phenomenon [S. Amari Bio. Cybernetics 1977], [C. Laing & W. Troy Phsica D 2003],

◉ Pigmentation pattern in animal skin [J.D. Murray, Springer 2003], [S. Kondo, J.T.B. 2017]

Typical modelings:

$u = u(x,t)$:some density, $K = K(x)$: an integral kernel,

Spatial convolution with suitable integral kernel $K * u = \int K(x-y)u(y,t)dy$

Normal type

$$u_t = K * u - bu + \cdots ,$$

Dispersal, Growth rate・・・

$b > 0$: a const.

Advective type

$$u_t = -\nabla \cdot (u\nabla(K * u)) + \cdots$$

Velocity

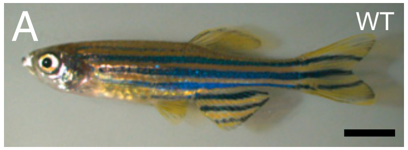
[c.f. Ninomiya, T., Yamamoto, J.M.B 2017, J.J.I.A.M., 2018]

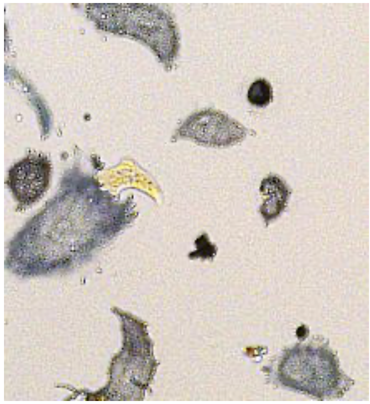
Biological examples of nonlocal interactions

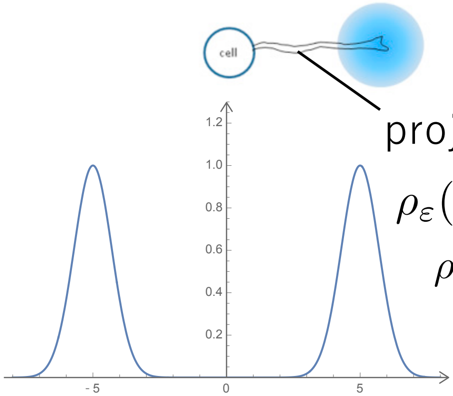
◉ Cell size and cell projections

▪Pigment cells in skin of zebra fish

[Yamanaka, Kondo, P.N.A.S., 2009] [Yamanaka, Kondo, P.N.A.S. 2009] [Kondo. J. Theor. Biol., 2017]







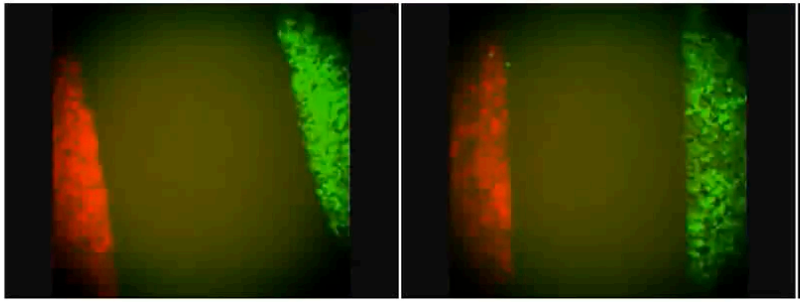
$\rho_\varepsilon(|x|-l),$

$\rho_\varepsilon \in C_0^\infty,$

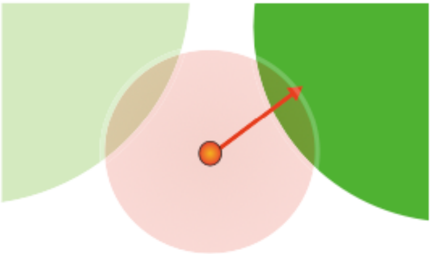
▪HEK 293 cells for experiments of cell adhesions

[Togashi et al. J. Cell Biol. 2016] [Murakawa, Togashi, J.T.B., 2015]

Movies of cell migration and adhesions:



Shematic figure of sensing function:



Size of cell body

II

Sensing function

Mathematical models with advective non-local interactions

① Aggregation diffusion model (Collective motion & cell migration)

[Carrillo, Craig, Yao, Model. Simul. Sci. Eng. Technol., 2019]

$$\rho_t = \Delta\rho^m - \underbrace{\nabla\cdot(\rho\nabla(K*\rho))}_{\text{advection}}, \quad m \geq 1$$

$m=1$ → linear diffusion ρ_{xx} : Nonlocal Fokker–Planck equation

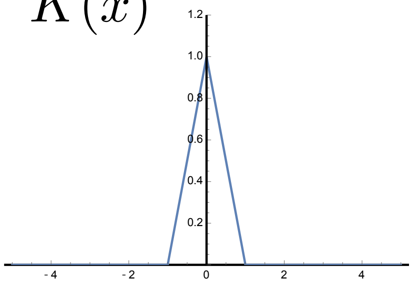
② Cell adhesion model [Togashi, Murakawa, J.T.B., 2015], [Carrillo, Murakawa, Sato, Togashi, Trush, J.T.B., 2019]

$$\frac{\partial u}{\partial t} = \Delta u^2 - \underbrace{\nabla\cdot(u(1-u)\nabla(K*u))}_{\text{advection}} + f(u),$$

$K(\boldsymbol{x}) = (R-|\boldsymbol{x}|)\chi_{B(0,R)}(\boldsymbol{x})$, $R>0$: Size of cell body,

$B(0,R)$: Ball, $\chi_{B(0,R)}(x) = \begin{cases} 1 & \text{if } x \in B(0,R), \\ 0 & \text{otherwise} \end{cases}$

$K(x)$

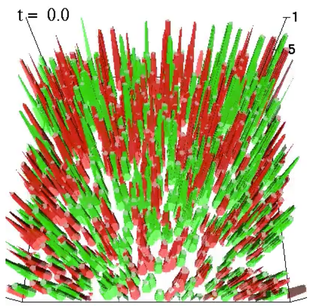


Motivation & Aim

Partial engulfment

◉ Motivation: To analyze (multiple components) cell adhesion model

$$\frac{\partial u}{\partial t} = \underbrace{\Delta u^2}_{\text{Porous medium type}} - \underbrace{\nabla \cdot (u(1-u)\nabla(W * u))}_{\text{advection}} + f(u)$$



- Difficulties: Nonlinear diffusivity & nonlocality

- Approximation for nonlinear diffusion by linear diffusion
[Murakawa, J.J.I.A.M., 2018]

◉ Aim: As a first step, we reveal whether the advective nonlocal interaction in the nonlocal Fokker–Planck equation can be approximated by a Keller–Segel system or not

Nonlocal Fokker–Planck equation

$$\rho_t = \rho_{xx} - \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x}(W * \rho)\right)$$

Model: Nonlocal Fokker–Planck equation

We analyze the following **nonlocal Fokker–Planck** equation:

velocity

$$(\mathrm{P}) \quad \begin{cases} \dfrac{\partial \rho}{\partial t} = \dfrac{\partial^2 \rho}{\partial x^2} - \dfrac{\partial}{\partial x}\left(\rho \dfrac{\partial}{\partial x}(W * \rho)\right), & t > 0,\ x \in \Omega := [-L, L], \\ \rho(x,0) := \rho_0(x) \in C^2(\Omega), \end{cases}$$

where periodic B.C. is imposed and $W * u = \int_{-L}^{L} W(x-y)u(y,t)dy$ for

2L-periodic $W \in L^1(\Omega)$.

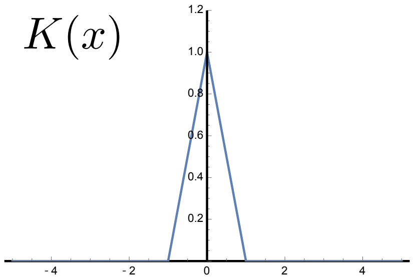
◉ Typical examples: $K(x) = (R - |x|)\chi_{B(0,R)}(x),\quad k_j(x) = \dfrac{1}{2\sqrt{d_j}\sinh \frac{L}{\sqrt{d_j}}} \cosh \dfrac{L - |x|}{\sqrt{d_j}},$

$B(0,R)$: a ball with radius R , $\chi_B(x)$: characteristic func.

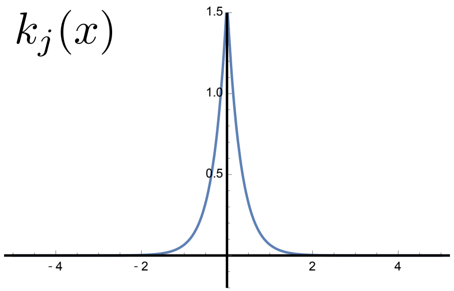
◉ $(W * \rho)_x = W * \rho_x$ is the velocity of advection term.

Examples of $W(x)$:

$K(x)$



$k_j(x)$



Keller–Segel system for approximation

We introduce the auxiliary attractive and repulsive substances $v_j^\varepsilon(x,t),\ (j=1,\cdots,M)$ in the part of advective nonlocal interaction.

[c.f. Ninomiya, T., Yamamoto, J.M.B 2017, J.J.I.A.M., 2018]

$$
(\mathrm{KS}_\varepsilon)\quad
\begin{cases}
\rho_t^\varepsilon = \rho_{xx}^\varepsilon - \dfrac{\partial}{\partial x}\left(\rho^\varepsilon \dfrac{\partial}{\partial x}\displaystyle\sum_{j=1}^{M} a_j v_j^\varepsilon\right),\\
(v_j^\varepsilon)_t = \dfrac{1}{\varepsilon}(d_j(v_j^\varepsilon)_{xx} - v_j^\varepsilon + \rho^\varepsilon),\ (j=1,\dots,M),\\
\rho^\varepsilon(x,0)=\rho_0(x),\ v_j^\varepsilon(x,0)=(v_j^\varepsilon)_0(x)\in C^2(\Omega)
\end{cases}
$$

where $0<\varepsilon\ll 1,\ \ d_j>0$, and $\{a_j\}_{j=1}^M$ are nonzero constants.

◉ If $M=1$, $(\mathrm{KS}_\varepsilon)$ becomes the **Keller–Segel eq**. with linear sensitive func.

◉ Taking the limit of $\varepsilon\to 0$, we expect that $0=d_j(v_j^\varepsilon)_{xx}-v_j^\varepsilon+\rho^\varepsilon$.

Base of non-local interaction

In fact, we have $v_j^\varepsilon(x,t)=(k_j*\rho^\varepsilon)(x,t)$, where $k_j(x):=\dfrac{1}{2\sqrt{d_j}\sinh\frac{L}{\sqrt{d_j}}}\cosh\dfrac{L-|x|}{\sqrt{d_j}}$.

➡ (P) with $W(x)=\displaystyle\sum_{j=1}^{M}a_jk_j(x)$

Main result 1: singular limit & order estimate

Theorem 1:

Let M be an arbitrary fixed natural number, and $\rho(x,t)$ be a solution of (P) equipped with $W=\sum_{j=1}^{M} a_j k_j(x)$ and the initial value $\rho_0(x)\in C^2(\Omega)$, and $\rho^\varepsilon(x,t)$ be a solution of $(\mathrm{KS}_\varepsilon)$ equipped with

(1) $$\left(\rho^\varepsilon, v_1^\varepsilon, \ldots, v_M^\varepsilon\right)(x,0)=(\rho_0, k_1*\rho_0, \ldots, k_M*\rho_0)(x).$$

Then, for any $\varepsilon>0$ and $T>0$, there exist positive constants C_1 and C_2 which depend on a_j and T, but are independent of ε such that

$$\sup_{t\in[0,T]}\|\rho^\varepsilon(\cdot,t)-\rho(\cdot,t)\|_{C(\Omega)}\le C_1\varepsilon,$$
$$\sup_{t\in[0,T]}\left\|v_j^\varepsilon(\cdot,t)-k_j*\rho(\cdot,t)\right\|_{C(\Omega)}\le C_2\varepsilon.$$

A natural question

What is the relationship between **any even potential** and the Keller–Segel system?

$$
(\mathrm{KS}_\varepsilon) \qquad \begin{cases} \rho^\varepsilon_t = \rho^\varepsilon_{xx} - \dfrac{\partial}{\partial x}\left(\rho^\varepsilon \dfrac{\partial}{\partial x} \displaystyle\sum_{j=1}^{M} a_j v^\varepsilon_j\right), \\ (v^\varepsilon_j)_t = \dfrac{1}{\varepsilon}(d_j (v^\varepsilon_j)_{xx} - v^\varepsilon_j + \rho^\varepsilon), \ (j = 1, \ldots, M), \end{cases}
$$

Realization of any even kernel

Setting $d_j = \dfrac{1}{(j-1)^2}$, we have $k_j(x) = \dfrac{j-1}{2\sinh(j-1)L}\cosh(j-1)(L-|x|).$

We set d_1 is **sufficiently large.**

Theorem 2 (cf. Existence of $\{c_j\}_{j=0}^n$: [Ninomiya, T., Yamamoto, J.M.B., 2017])

Assume that $W(x)$ is even in Ω and in $C^n([0,L])$ and, let $f(x)$ be $f(x) := W(L-\log(x+\sqrt{x^2-1})) = W(\cosh^{-1}(L-x))$. Then, for any $n \in \mathbb{N}$ there exist explicit $\{c_j\}_{j=0}^n$ such that

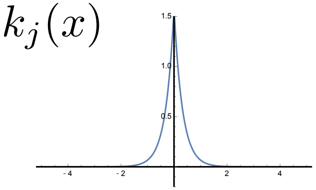
$$\sup_{x\in[0,L]}\left|W(x) - \sum_{j=0}^{n} c_j \cosh j(L-x)\right| \leq \frac{1}{2^n(n+1)!}\left(\frac{\cosh L - 1}{2}\right)^{n+1} \max_{y\in[1,\cosh L]} |f^{(n)}(y)|.$$

From this theorem, we can approximate the solution of (P) with **any kernel** by solution of $(\mathrm{KS}_\varepsilon)$ **with specified parameters**.

We set $a_j = \begin{cases} 2Lc_0 & (j=1), \\ 2c_{j-1}\sinh((j-1)L)/(j-1) & (j=2,\dots M) \end{cases}$ for $k_j(x)$.

Realization of arbitrary kernel

$k_j(x)$



Theorem 3:

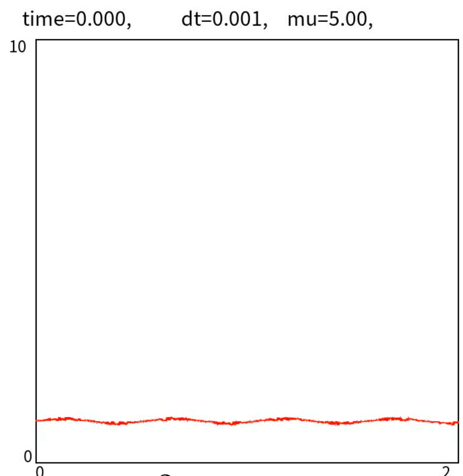
For any even 2L−periodic $C^n([0, L])$ function W, any $\varepsilon > 0$ and any $T > 0$, there exist a Keller−Segel system $(\mathrm{KS}_\varepsilon)$ with $M+1$ component, and a positive constant C_T independent of ε such that

$$\sup_{t\in[0,T]} \|\rho^\varepsilon(\cdot,t) - \rho(\cdot,t)\|_{L^2(\Omega)} \le C_T\varepsilon$$

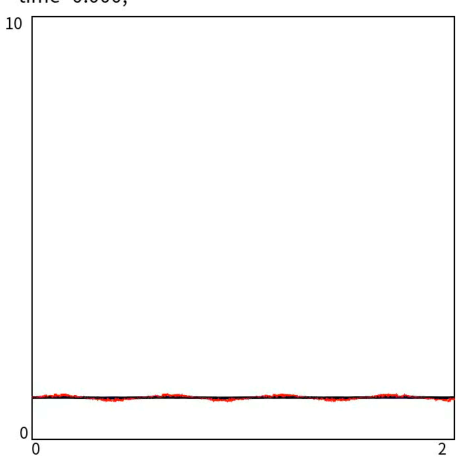
where ρ is the solution of (P) equipped with $\rho_0(x) \in C^2(\Omega)$ and ρ^ε is the first component of $(\mathrm{KS}_\varepsilon)$ equipped with (1).

Numerical experiments:

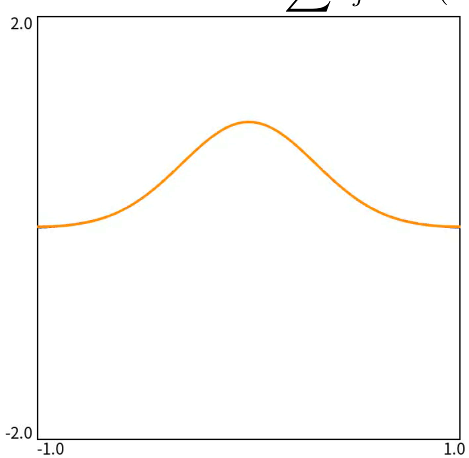
(P)



$(\mathrm{KS}_\varepsilon)$



Profiles of W and $\sum c_j \cosh(L - |x|)$



$W(x) = e^{-5x^2}, \ \rho_0(x) = 1 + \xi(x), \ \mu = 5.0, \ d_1 = 1000000, \ M = 7$

Concluding remarks

◉ Combining the solution of a Keller–Segel system with the linear sensitive function $(\mathrm{KS}_\varepsilon)$ can approximate the solution of the nonlocal Fokker–Planck eq. with any integral kernel in L^2 .

$$(\mathrm{P}) \qquad\qquad\qquad\qquad (\mathrm{KS}_\varepsilon)$$

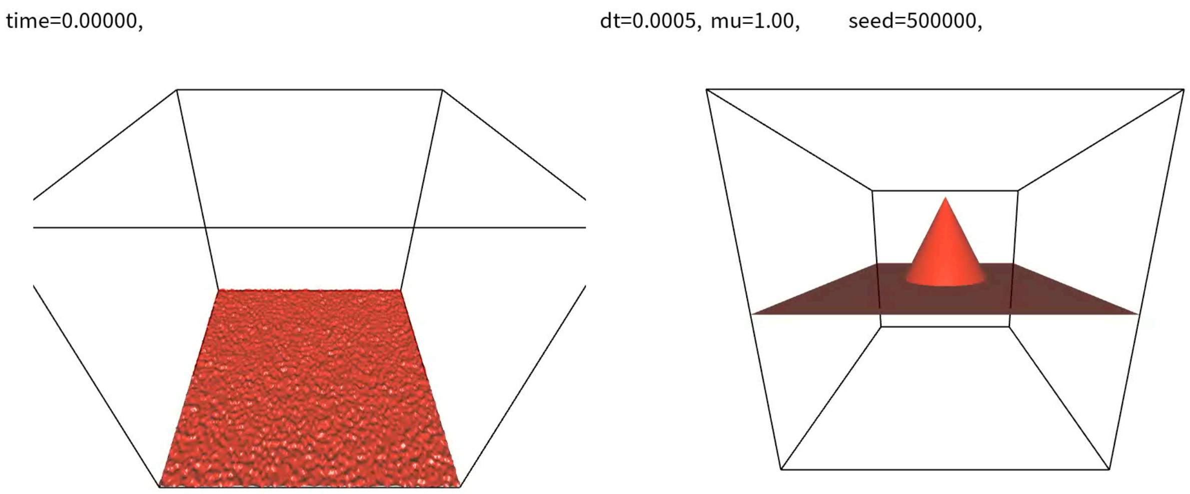
$$\rho_t = \rho_{xx} - \frac{\partial}{\partial x}\Big(\rho \frac{\partial}{\partial x}(W * \rho)\Big) \quad \underset{\varepsilon \to 0+0}{\approx} \quad \begin{cases} \rho^\varepsilon_t = \rho^\varepsilon_{xx} - \dfrac{\partial}{\partial x}\Big(\rho^\varepsilon \dfrac{\partial}{\partial x}\displaystyle\sum_{j=1}^{M} a_j v^\varepsilon_j\Big), \\ (v^\varepsilon_j)_t = \dfrac{1}{\varepsilon}(d_j (v^\varepsilon_j)_{xx} - v^\varepsilon_j + \rho^\varepsilon), \ (j = 1, \dots, M), \end{cases}$$

◉ The time of numerical simulation of $(\mathrm{KS}_\varepsilon)$ is **shorter than** that of (P).

(Typical scheme of numerical integration was applied in (P).)

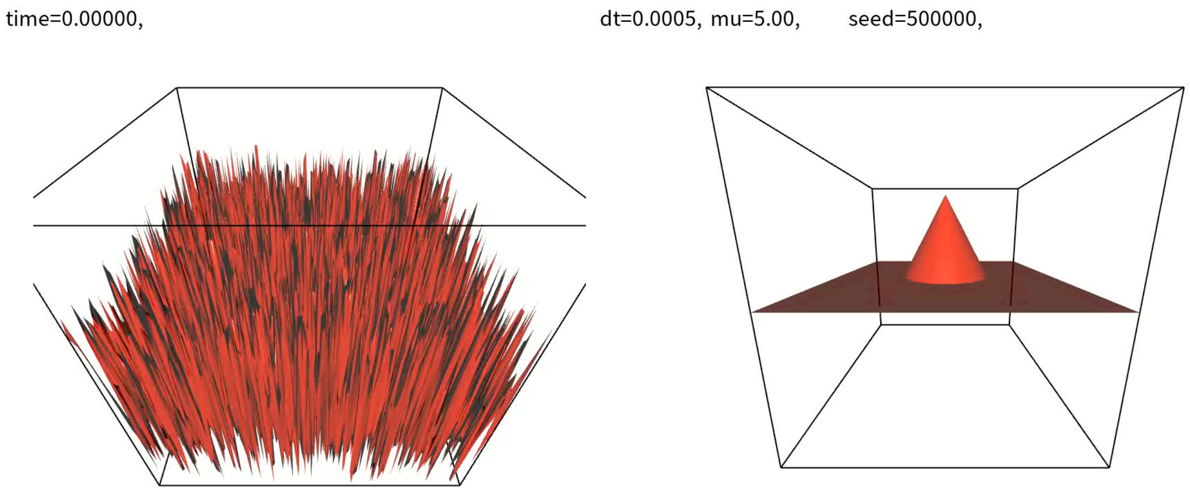
◉ We will extend our theory to case of higher dimension and the model of cell adhesion.

Numerical results in 2D



(P)

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2} - \mu \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x}(W * \rho)\right),$$



$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2} - \mu \frac{\partial}{\partial x}\left(\rho \boxed{(1-\rho)} \frac{\partial}{\partial x}(W * \rho)\right),$$

volume filling effect

"Spreading fronts arising from the singular limit of reaction-diffusion systems"

Chang-Hong Wu (National Yang Ming Chiao Tung University, Taiwan)

To gain insight into the formation of spreading fronts of invasive species, in this presentation we will focus on the singular limit of reaction-diffusion systems.
We investigate the dynamics of the limiting systems and give some interpretations for spreading fronts from the modeling viewpoint. The talk is based on joint works with Hirofumi Izuhara and Harunori Monobe.

Spreading fronts arising from the singular limit of reaction-diffusion systems

Chang-Hong Wu

National Yang Ming Chiao Tung University

Based on joint works with Hirofumi Izuhara and Harunori Monobe

Reaction-diffusion systems: from the past to the future
dedicated to the memory of Professor Masayasu Mimura
2023.10.31-11.2 ICMMA, Meiji University

Outline

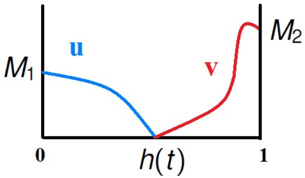
- Introduction
- Spatial segregation limit
- Numerical results
- Summary

The habitat segregation phenomenon

- Mimura-Yamada-Yotsutani (1985,1986,1987)

$$\begin{cases} u_t = d_1 u_{xx} + f(u), \ 0 < x < h(t), \ t > 0, \\ v_t = d_2 v_{xx} + g(v), \ h(t) < x < 1, \ t > 0, \\ u(0,t) = M_1, \ v(1,t) = M_2, \ t > 0, \\ u(h(t),t) = v(h(t),t) = 0, \ t > 0, \\ h'(t) = -\mu_1 u_x(h(t),t) - \mu_2 v_x(h(t),t), \ t > 0, \\ h(0) = h_0 \in (0,1), \\ u(x,0) = u_0(x), \ 0 < x < h_0, \ v(x,0) = v_0(x), \ h_0 < x < 1. \end{cases}$$

- The global existence, uniqueness, regularity, asymptotic behavior of solutions, and stability of stationary solutions



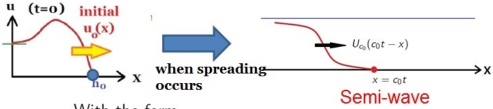
- Classical Stefan problems arise in many physical problems such as the melting of materials and freezing of liquid (Rubinstein 1971)

The spreading of an invasive species

- Du-Lin (2010):

$$\begin{cases} u_t = du_{xx} + u(a - bu), \ 0 < x < h(t), \ t > 0, \\ u_x(0,t) = 0, \ u(h(t),t) = 0, \ t > 0, \\ h'(t) = -\mu u_x(h(t),t), \ t > 0, \\ h(0) = h_0, \ u(x,0) = u_0(x), \ 0 < x < h_0, \end{cases}$$

- Spreading-Vanishing dichotomy
  - critical length: $h^* := \frac{\pi}{2}\sqrt{\frac{d}{a}}$
- Semiwaves exist when $c_0 \in (0, 2\sqrt{ad})$
- The global dynamics (Du-Matsuzawa-Zhou 2014, 2015)
  - If $h_\infty := \lim_{t\to\infty} h(t) = \infty$, then $u(x,t) \to U_{c_0}(h(t) - x)$ uniformly in x, as $t \to \infty$.
- A biological explanation for FBC: population loss (Bunting-Du-Krakowski 2012)



With the form

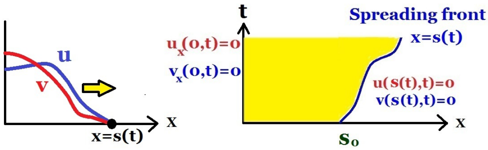
$$h(t) = ct, u(x,t) = U(ct - x),$$

$$cU' = dU'' + U(a - bU) \ in \ (0, \infty)$$
$$U(0) = 0, \ U(\infty) = a/b \qquad U'_{c_0}(0) = c_0/\mu.$$

- Scalar equations: see also Kaneko-Yamada (2011), Bunting-Du-Krakowski (2012), Chang-Chen (2012), Du-Matano-Wang (2014), Du-Lou (2015), Du-Matsuzawa-Zhou (2014,2015), Kaneko-Matsuzawa (2015,2018), Wang (2015,2016), Monobe-W. (2016), Zhao-Wang (2018), Kaneko-Matsuzawa-Yamada (2020,2022,2023), El-Hachem-McCue-Simpson (2021) and more.

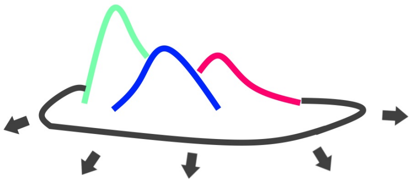
- Guo-W. (2012): two species with the weak competition

$$\begin{cases} u_t = u_{xx} + u(1-u-kv), \ 0<x<s(t), \ t>0, \\ v_t = Dv_{xx} + rv(1-v-hu), \ 0<x<s(t), \ t>0, \\ u_x(0,t) = v_x(0,t) = 0, \ u(s(t),t) = v(s(t),t) = 0, \ t>0, \\ s'(t) = -\mu_1 u_x(s(t),t) - \mu_2 v_x(s(t),t), \ t>0, \end{cases}$$



- See also Guo-W. (2015), W. (2015, 2019), Du-W. (2018,2022), Wang-Zhang (2017), Liu-Huang-Wang (2019) for models with two different spreading fronts.

- Question: can reaction-diffusion systems approximate these problems?
- Fast-reaction limit, spatial segregation limit: Hilhorst-van der Hout-Peletier (1996), Dancer-Hilhorst-Mimura-Peletier (1999), Ei-Ikota-Mimura (1999), Hilhorst-Iida-Mimura-Ninomiya (2001), Hilhorst-Mimura-Schatzle (2003), Crooks-Dancer-Hilhorst-Mimura-Ninomiya (2004), Crooks-Dancer-Hilhorst (2007), Alfaro-Hilhorst-Matano (2008), Hilhorst-Mimura-Ninomiya (2009), Murakawa-Ninomiya (2011) and more



Competition models with a small parameter

Izuhara-Monobe-W. (2023):

$$(P_\varepsilon)\begin{cases}\partial_t u_i = d_i\Delta u_i + f_i(u_i) - \displaystyle\sum_{j=1,\ j\neq i}^{n} h_{ij}(u_i,u_j) - \frac{1}{\varepsilon}F_i(u_i,w), & \text{in } Q_T,\\ \partial_t w = d_w\Delta w + g(w) - \displaystyle\sum_{i=1}^{n}\frac{\beta_i}{\varepsilon}F_i(u_i,w), & \text{in } Q_T,\\ \partial_\nu u_i = \partial_\nu w = 0, & \text{on } \partial Q_T,\\ (u_i(x,0), w(x,0)) = (u_{i,0}(x), w_0(x)), & \text{in } \Omega,\end{cases}$$

- $i = 1,\cdots,n$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $Q_T := \Omega\times(0,T]$, $\partial Q_T := \partial\Omega\times(0,T]$.
- $d_i > 0$, $\beta_i > 0$, $\varepsilon > 0$ and $d_w \geq 0$ (if $d_w = 0$, $\partial_\nu w = 0$ is dropped)
- f_i and g stand for the intraspecific growth functions of u_i and w, respectively;
- h_{ij} measures the interspecific competition from the species u_j to species u_i;
- F_i measures the interspecific competition between the species u_i and w.
- Izuhara-Monobe-W. (2021) considered $n = 2$, $g \equiv 0$ and $d_w = 0$.

Competition models with a small parameter

- In this talk, we consider

$$f_i(u_i) = r_i u_i\left(1 - \frac{u_i}{K_i}\right), \quad h_{ij} = \alpha_{ij} u_i u_j, \quad F_i(u_i, w) = \gamma_i u_i w, \quad g(w) = r_w w\left(1 - \frac{w}{K_w}\right)$$

- For initial data, $0 \leq u_{i,0} \leq K_i$, $0 \leq w_0 \leq K_w$ and $u_{i,0} \in C(\overline{\Omega})$,

$$w_0 \in C(\overline{\Omega}) \text{ if } d_w > 0; \; w_0 \in L^\infty(\Omega) \text{ if } d_w = 0.$$

Proposition

For any $T > 0$ and $\varepsilon > 0$, there exists a unique solution $(u_i^\varepsilon, w^\varepsilon)$ to (P_ε) with the following regularity

(i) *if $d_w > 0$,*

$$u_i^\varepsilon, w^\varepsilon \in C(\overline{Q_T}) \cap C^{2,1}(\overline{\Omega} \times (0, T])$$

(ii) *if $d_w = 0$,*

$$u_i^\varepsilon \in C(\overline{Q_T}) \cap C^1\Big((0, T]; C(\overline{\Omega})\Big) \cap C\Big((0, T]; W^{2,p}(\Omega)\Big), \; w^\varepsilon \in C^1([0, T]; L^\infty(\Omega))$$

for $i = 1, 2, \cdots, n$ and $Q_T := \Omega \times (0, T]$. Moreover,

$$0 \leq u_i^\varepsilon \leq K_i, \; i = 1, ..., n, \quad \text{and} \quad 0 \leq w^\varepsilon \leq K_w.$$

The case $d_w = 0$

We first consider $d_w = 0$:

$$
(P_\varepsilon)\begin{cases}
\partial_t u_i = d_i \Delta u_i + f_i(u_i) - \sum_{j=1,\ j\neq i}^{n} h_{ij}(u_i, u_j) - \frac{1}{\varepsilon} F_i(u_i, w), & \text{in } Q_T,\\
\partial_t w = g(w) - \sum_{i=1}^{n} \frac{\beta_i}{\varepsilon} F_i(u_i, w), & \text{in } Q_T,\\
\partial_\nu u_i = 0, & \text{on } \partial Q_T,\\
(u_i(x,0), w(x,0)) = (u_{i,0}(x), w_0(x)), & \text{in } \Omega,
\end{cases}
$$

- Claim: there exist $\hat{u}_i$, $\hat{w} \in L^2(0, T; H^1(\Omega))$ such that

$$u_i^\varepsilon \to \hat{u}_i, \quad w^\varepsilon \to \hat{w} \quad \text{strongly in } L^2(Q_T) \text{ and weakly in } L^2(0, T; H^1(\Omega))$$

  as $\varepsilon \to 0$ (up to a subsequence) and

$$\hat{u}_i \hat{w} = 0 \quad \text{a.e. in } Q_T\ \forall i \quad \text{(Segregation property)}$$

- Multiplying the equation of u_i by β_i and sum them up for i:

$$\sum_{i=1}^{n} \beta_i \partial_t u_i = \sum_{i=1}^{n} \Big[\beta_i d_{u_i} \Delta u_i + \beta_i f_i(u_i) - \sum_{j=1,\ j\neq i}^{n} \beta_i h_{ij}(u_i, u_j)\Big] - \sum_{i=1}^{n} \frac{\beta_i}{\varepsilon} F_i(u_i, w).$$

- We subtract the equation of w from the above equation,

$$\sum_{i=1}^{n} \beta_i \partial_t u_i - \partial_t w = \sum_{i=1}^{n} \Big[\beta_i d_i \Delta u_i + \beta_i f_i(u_i) - \sum_{j=1,\ j\neq i}^{n} \beta_i h_{ij}(u_i, u_j)\Big] - g(w).$$

- We multiply it by a test function ζ. Then, by integrating it over Q_T, using integration by parts,

$$\iint_{Q_T} \Big(\sum_{i=1}^n \beta_i u_i - w\Big)\zeta_t - \sum_{i=1}^n \Big[\beta_i d_i \nabla u_i \cdot \nabla\zeta + \beta_i f_i(u_i)\zeta - \sum_{j=1,\ j\neq i}^n \beta_i h_{ij}(u_i, u_j)\zeta\Big]$$
$$- \iint_{Q_T} g(w)\zeta = -\int_\Sigma \Big(\sum_{i=1}^n \beta_i u_{i,0} - w_0\Big)\zeta(x,0),$$

where $(u_i, w) = (u_i^\varepsilon, w^\varepsilon)$.

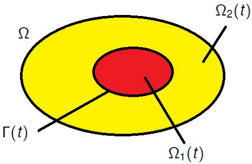
- Passing to the limit in ε along a subsequence,

$$\iint_{Q_T} \Big(\sum_{i=1}^n \beta_i \hat{u}_i - \hat{w}\Big)\partial_t\zeta - \Big(\sum_{i=1}^n \beta_i d_{u_i} \nabla \hat{u}_i\Big) \cdot \nabla\zeta \tag{2.1}$$
$$+ \Big[\sum_{i=1}^n \beta_i f_i(\hat{u}_i) + \sum_{i=1}^n \sum_{j=1, j\neq i}^n \beta_i h_{ij}(\hat{u}_i, \hat{u}_j) - \hat{w} g(\hat{w})\Big]\zeta \, dxdt$$
$$= -\int_\Omega \Big(\sum_{i=1}^n \beta_i u_{i,0} - w_0\Big)\zeta(x,0)\, dx$$

for any $\zeta \in C^\infty(\overline{Q_T})$ with $\zeta(x, T) = 0$.

- Assume that $\exists\Omega_1(t)$, $\Omega_2(t)$ and $\Gamma(t)$ are 'good enough':

$$\begin{cases} \Omega_1(t) := \{x \in \Omega |\ \hat{u}_i(x,t) > 0,\ i = 1,2,...,n\}, \\ \Omega_2(t) := \{x \in \Omega |\ \hat{w}(x,t) > 0\}, \\ \overline{\Omega} = \overline{\Omega}_1(t) \cup \overline{\Omega}_2(t), \quad \Gamma(t) := \overline{\Omega}_1(t) \cap \overline{\Omega}_2(t), \end{cases}$$



- Next, we define

$$Q_T^1 := \bigcup_{0<t\le T} \Omega_1(t) \times \{t\}, \quad Q_T^2 := \bigcup_{0<t\le T} \Omega_2(t) \times \{t\}, \quad \Gamma_T := \bigcup_{0<t\le T} \Gamma(t) \times \{t\}.$$

- By some assumptions on the smoothness of interfaces, we can separate Q_T into Q_T^1 and Q_T^2, respectively, in the integral (2.1), we have

$$\iint_{Q_T^1} \zeta \sum_{i=1}^n \beta_i \Big[-\partial_t \hat{u}_i + d_{u_i} \Delta \hat{u}_i + f_i(\hat{u}_i) - \sum_{j=1, j\ne i}^n h_{ij}(\hat{u}_i, \hat{u}_j) \Big]\, dxdt$$

$$+ \iint_{Q_T^2} \zeta(\partial_t \hat{w} - g(\hat{w}))\, dxdt - \int_0^T \int_{\Gamma(t)} \zeta \Big\{ \hat{w} V + \sum_{i=1}^n \beta_i d_{u_i} \partial_\nu \hat{u}_i \Big\}\, d\Omega dt, \qquad (2.2)$$

for any $\zeta \in C^\infty(\overline{Q_T})$ with $\zeta(x,T) = 0$, where V denotes the normal velocity from $\Omega_1(t)$ to $\Omega_2(t)$ at $\Gamma(t)$ and we further assume that initial conditions are segregated.

Main result ($d_w = 0$)

Theorem (Izuhara-Monobe-W. 2023)

Assume that $d_w = 0$. Suppose that $\Gamma(t)$ is a smooth closed oriented hypersurface satisfying $\partial\Omega \cap \partial\Omega_1(t) = \emptyset$ for all $t \in [0, T]$, and $\Gamma(t)$ moves smoothly. Moreover, we suppose that $\hat{w}$ is also smooth in $\overline{Q_T^2}$ and $\hat{u}_i$ $(i = 1, 2, ..., n)$ is smooth in $\overline{Q_T^1}$. Then $\hat{w}$ satisfies

$$\frac{d}{dt}\hat{w}(x,t) = g(\hat{w}), \quad (x,t) \in Q_T^2,$$

and $(\hat{u}_i, \Omega_i(t), \Gamma(t))$ satisfies the following free boundary problem:

$$\begin{cases} \partial_t u_i = d_i \Delta u_i + f_i(u_i) - \displaystyle\sum_{j=1,\ j\neq i}^{n} h_{ij}(u_i, u_j) & \text{in } Q_T^1,\ i = 1, 2, \cdots, n, \\ u_i = 0 & \text{on } \Gamma_T,\ i = 1, 2, \cdots, n, \\ \hat{w}(x,t)V = -\displaystyle\sum_{i=1}^{n} \beta_i d_i \partial_\nu u_i & \text{on } \Gamma_T, \\ w(x,t) = w_0(x) & \text{in } Q_T^2, \\ u(x,0) = u_{i,0}(x) & \text{in } \Omega_1(0),\ i = 1, 2, \cdots, n, \\ w(x,0) = w_0(x) & \text{in } \Omega_2(0), \end{cases} \tag{2.3}$$

where

$$Q_T^1 := \bigcup_{0<t\leq T} \Omega_1(t) \times \{t\}, \quad Q_T^2 := \bigcup_{0<t\leq T} \Omega_2(t) \times \{t\}, \quad \Gamma_T := \bigcup_{0<t\leq T} \Gamma(t) \times \{t\}.$$

Proposition (Izuhara-Monobe-W. 2023)

Assume that $\Omega = (-L, L)$ *and*

$$\begin{cases} s_{\pm}(0) = \pm s_0, \ u_i(x,0) = u_{i,0}(x), \ x \in [-s_0, s_0] \text{ for some } s_0 \in (0, L); \\ u_{i,0} \in C^2([-s_0, s_0]), \ u_{i,0}(\pm s_0) = 0, \ w_0 \in C^\alpha([-L, L]), \ \alpha \in (0,1), \\ 0 \le u_{i,0}(x) \le K_i, \ x \in [-s_0, s_0]; \ 0 < w_0(x) \le K_w, \ x \in [-L, L], \end{cases}$$

Then the free boundary problem (2.3) *admits a unique classical solution*

$$(u_i, s) \in [C^{2+\alpha, 1+\frac{\alpha}{2}}(D_T)]^n \times C^{1+\frac{\alpha}{2}}([0, T]), \ i = 1, 2, \cdots, n,$$

for some small $T > 0$*, where*

$$D_T := \{(x,t) | \, s_-(t) \le x \le s_+(t), \ t \in (0, T]\}.$$

Moreover, the unique solution can be extended up to a time T^* *satisfying either* $\lim_{t \nearrow T^*} s_+(t) = L$ *or* $\lim_{t \nearrow T^*} s_-(t) = -L$.

- Since $w_0 > 0$ in $[-L, L]$ and w satisfies the logistic equation, we have $k_i > 0$ such that $k_1 \le w \le k_2$ for all $t \ge 0$.
- Contraction mapping principle

Main result ($d_w > 0$)

Using a similar argument as in $d_w = 0$, we can obtain

Theorem (Izuhara-Monobe-W. 2023)

Assume that $d_w > 0$. Suppose that $\Gamma(t)$ is a smooth closed oriented hypersurface satisfying $\partial\Omega \cap \partial\Omega_1(t) = \emptyset$ for all $t \in [0, T]$, and $\Gamma(t)$ moves smoothly. Moreover, we suppose that $\hat{w}$ is also smooth in $\overline{Q_T^2}$ and $\hat{u}_i$ $(i = 1, 2, ..., n)$ is smooth in $\overline{Q_T^1}$. Then $(\hat{u}_i, \hat{w}, \Omega_i(t), \Gamma(t))$ satisfies the following free boundary problem:

$$\begin{cases} \partial_t u_i = d_{u_i}\Delta u_i + f_i(u_i) - \sum_{j=1,\ j\neq i}^{n} h_{ij}(u_i, u_j) & \text{in } Q_T^1, \\ \partial_t w = d_w \Delta w + g(w) & \text{in } Q_T^2, \\ u_i = w = 0 & \text{on } \Gamma_T, \\ d_w \partial_\nu w + \sum_{i=1}^{n} \beta_i d_{u_i} \partial_\nu u_i = 0 & \text{on } \Gamma_T, \\ \partial_\nu w = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_{i,0}(x) & \text{in } \Omega_1(0), \\ w(x, 0) = w_0(x) & \text{in } \Omega_2(0), \end{cases} \tag{2.4}$$

- When $n = 1$, the free boundary problem reduces to the one proposed by Dancer-Hilhorst-Mimura-Peletier (1999).

The revisit: the role of diffusion d_1

We revisit the free boundary problem studied by Du and Lin (2010).

- Consider

$$\begin{cases} u_t = d_1 u_{xx} + u(a - bu) - \dfrac{b_{13}}{\varepsilon} uw, & x \in \Omega,\ t > 0, \\ w_t = r_w w(1 - \dfrac{w}{K_w}) - \dfrac{b_{31}}{\varepsilon} uw, & x \in \Omega,\ t > 0, \\ \partial_\nu u = 0, \quad x \in \partial\Omega,\ t > 0, \end{cases}$$

  where $u_0 > 0$ and $w_0 > 0$ are spatial segregated.

- Consider 1D case and spatial symmetry, we are led to study

$$\begin{cases} u_t = d_1 u_{xx} + u(a - bu), & 0 < x < h(t),\ t > 0, \\ u_x(0,t) = 0,\ u(h(t),t) = 0, & t > 0, \\ h'(t) = -\dfrac{d_1 b_{31}}{w(h(t),t) b_{13}} u_x(h(t),t), & t > 0, \\ h(0) = h_0,\ u(x,0) = u_0(x), & 0 < x < h_0, \end{cases}$$

- Q: the role of d_1 in the spreading of u? We can revisit the free boundary problems in the existing literature.

The revisit: two weakly competitive species

A modeling perspective for the free boundary condition in Guo-W. (2012):

- Consider

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1-u-hv) - \dfrac{b_{13}}{\varepsilon} uw, & x \in \Omega,\ t > 0, \\ v_t = d_2 v_{xx} + r_2 v(1-v-ku) - \dfrac{b_{23}}{\varepsilon} vw, & x \in \Omega,\ t > 0, \\ w_t = r_w w(1 - \dfrac{w}{K_w}) - \dfrac{b_{31}}{\varepsilon} uw - \dfrac{b_{32}}{\varepsilon} vw, & x \in \Omega,\ t > 0, \\ \partial_\nu u = \partial_\nu v = 0, \quad x \in \partial\Omega,\ t > 0, \end{cases}$$

  where $u_0 > 0$ (resp., $v_0 > 0$) and $w_0 > 0$ are spatial segregated.

- Then the limiting problem has the free boundary condition:

$$h'(t) = -\frac{d_1 b_{31}}{w(h(t),t) b_{13}} u_x(h(t),t) - \frac{d_2 b_{32}}{w(h(t),t) b_{23}} v_x(h(t),t), \quad t > 0,$$

  where w obeys the logistic equation.

Numerical results

- Q: is there any difference between the cases $d_w > 0$ and $d_w = 0$?
- We consider the interaction between three weakly competing species u_i $(i = 1, 2, 3)$ and one strongly competing species w:

$$\partial_t u_1 = d_1 \partial_{xx} u_1 + r_1 u_1 \left(1 - \frac{u_1}{K_1}\right) - \alpha_{12} u_1 u_2 - \alpha_{13} u_1 u_3 - \frac{1}{\varepsilon}\gamma_1 u_1 w,$$
$$\partial_t u_2 = d_2 \partial_{xx} u_2 + r_2 u_2 \left(1 - \frac{u_2}{K_2}\right) - \alpha_{21} u_2 u_1 - \alpha_{23} u_2 u_3 - \frac{1}{\varepsilon}\gamma_2 u_2 w,$$
$$\partial_t u_3 = d_3 \partial_{xx} u_3 + r_3 u_3 \left(1 - \frac{u_3}{K_3}\right) - \alpha_{31} u_3 u_1 - \alpha_{32} u_3 u_2 - \frac{1}{\varepsilon}\gamma_3 u_3 w,$$
$$\partial_t w = d_w \partial_{xx} w + r_w w \left(1 - \frac{w}{K_w}\right) - \frac{1}{\varepsilon}\left(\beta_1\gamma_1 u_1 + \beta_2\gamma_2 u_2 + \beta_3\gamma_3 u_3\right) w,$$

- The parameter values are all fixed except for d_w and ε:

$$d_1 = 0.75,\ d_2 = 0.1,\ d_3 = 1.0,\ r_1 = 0.25,\ r_2 = 0.35,\ r_3 = 0.3,\ r_w = 0.25,$$
$$K_1 = 1.1,\ K_2 = 0.9,\ K_3 = 1.0,\ K_w = 1.0,\ \alpha_{12} = 0.15,\ \alpha_{13} = 0.25,\ \alpha_{21} = 0.2,$$
$$\alpha_{23} = 0.1,\ \alpha_{31} = 0.1,\ \alpha_{32} = 0.2,\ \gamma_1 = \gamma_2 = \gamma_3 = 1.0,\ \beta_1 = 0.8,\ \beta_2 = 0.75,\ \beta_3 = 0.7,$$

- The coexistence state: $(u_1^*, u_2^*, u_3^*) \approx (0.084760, 0.732021, 0.483733)$.

Numerical results

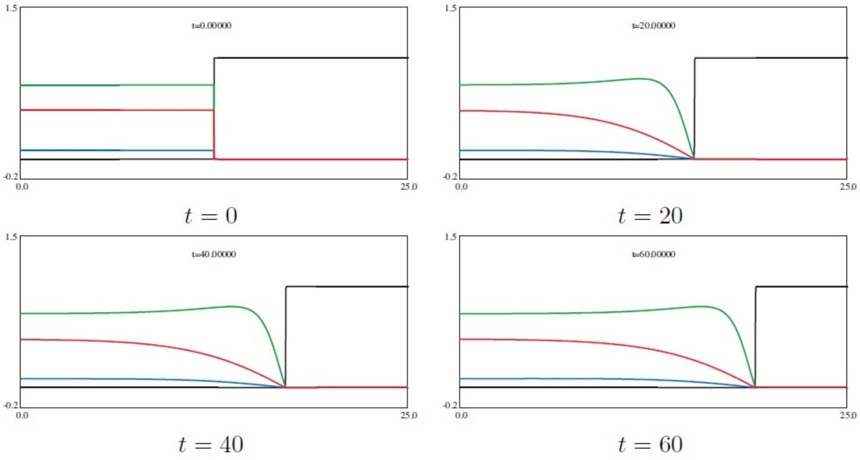


FIGURE 1. Snapshots of a solution behavior when $d_w = 0$, $\varepsilon = 0.00001$ and $L = 25$. The blue, green, red and black curves respectively mean u_1, u_2, u_3 and w, respectively.

Numerical results

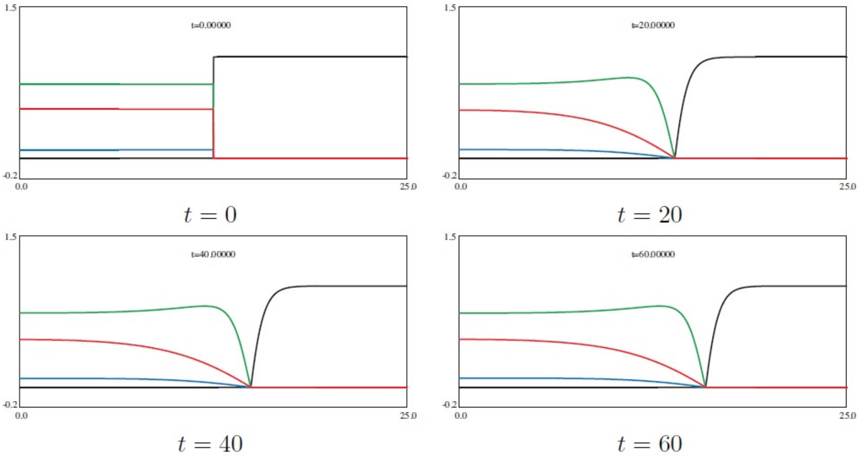


FIGURE 2. Snapshots of a solution behavior when $d_w = 0.1$, $\varepsilon = 0.00001$ and $L = 25$. The colors of curves indicate the same as the ones in Figure 1.

Numerical results

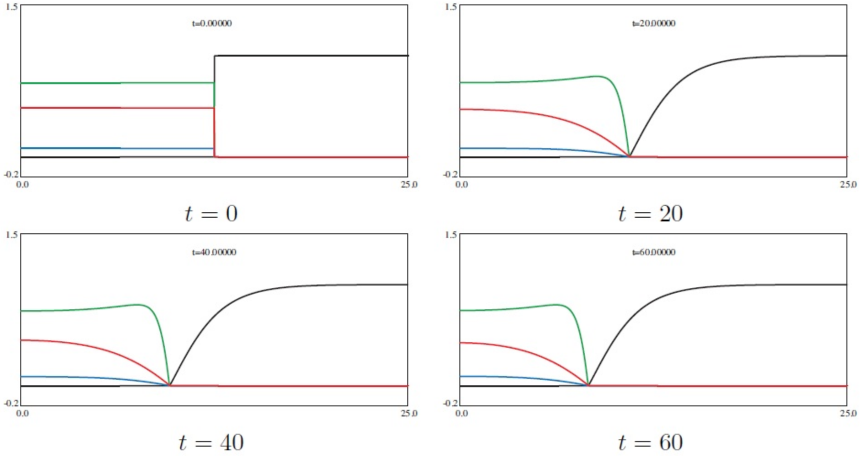


FIGURE 3. Snapshots of a solution behavior when $d_w = 0.7$, $\varepsilon = 0.00001$ and $L = 25$. The colors of curves indicate the same as the ones in Figure 1.

Numerical results

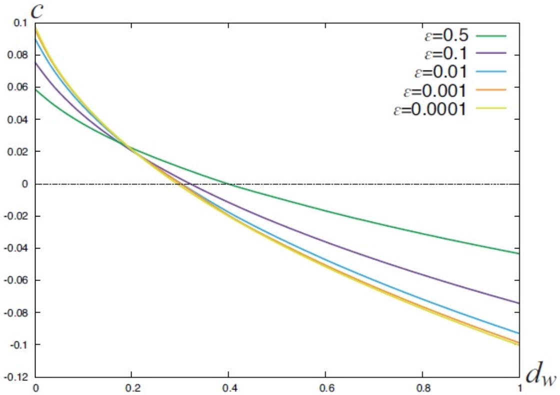
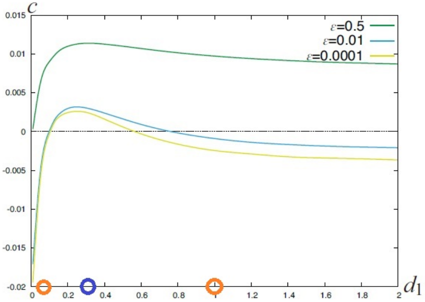
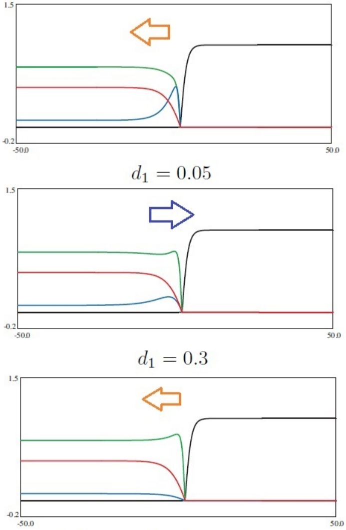


FIGURE 4. Relations between the diffusivity d_w and the traveling wave velocity c for each ε value are shown. The horizontal axis is d_w and the vertical one is c.

Numerical results



$d_w = 0.3.$



$d_1 = 0.05$

$d_1 = 0.3$

$\varepsilon = 0.0001 \qquad d_1 = 1.0$

Summary

- We consider the spatial segregation limit for two types of competition-diffusion systems and derive two free boundary problems.
- We revisit the free boundary problems in the literature and study the role of diffusion.
- We also present numerical results that demonstrate the complexity of the role played by diffusion rates.
- The interpretation of the parameter in the free boundary condition may be helpful for its application to real-world data (Izuhara-Monobe-W. 2021).

Thank you for your attention

"Segregation pattern in a reaction-diffusion model of asymmetric cell division"

Yoshihisa Morita (Ryukoku University, Japan)

We deal with a mathematical model describing polarity in the asymmetric cell division of C. elegans embryo. In the maintenance phase of asymmet- ric cell division anterior PAR protein (aPAR) and posterior PAR protein (pPAR) are exclusively formed and a segregation pattern is created for the polarizations of aPAR and pPAR. Seirin-Lee and Shibata (2015) proposed a 4-component reaction-diffusion system with mass conservation as a model to describe the segregation pattern. Later, some gradient-like dynamics and variational structure in a slightly modied model system were revealed by Morita and Serin-Lee (2021). In this talk we review their work and report a recent progress.

Segregation pattern in a reaction-diffusion model of asymmetric cell division

Yoshihisa Morita
Emeritus Professor of Ryukoku University
Joint Research Center for Science and Technology of Ryukoku Univ.

International Conference on
"Reaction-diffusion systems: from the past to the future"
— in memory of Prof. Masayasu Mimura —

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Contents

This lecture is based on the works by:

- M–Seirin-Lee (2021)

  S. Seirin-Lee (Kyoto University)

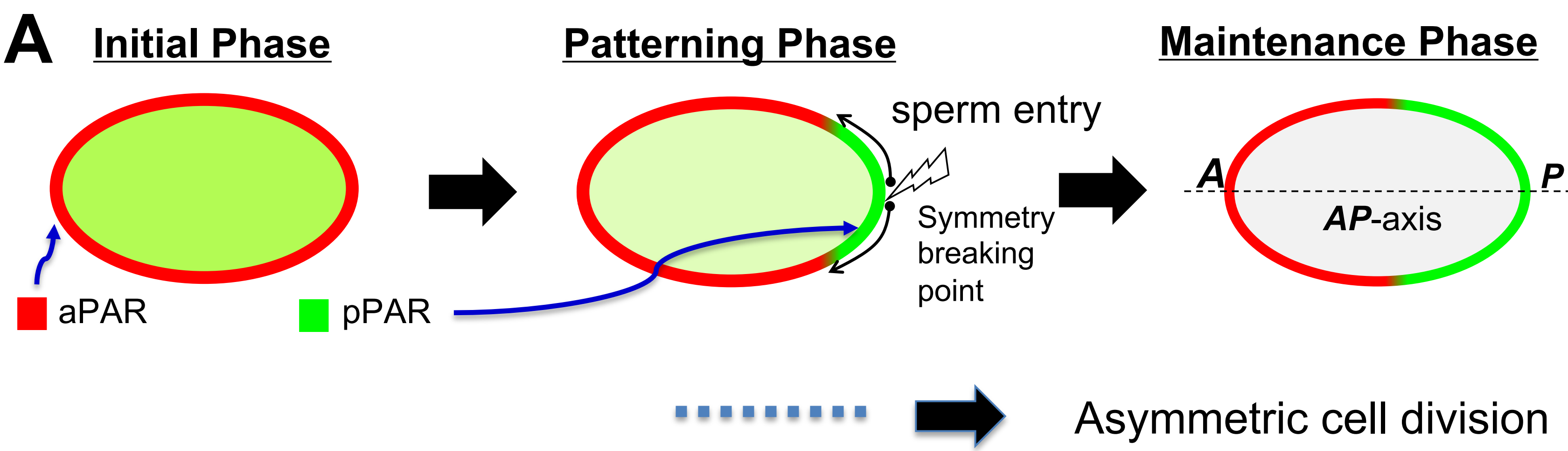
- M-Oshita (2023)

  Y. Oshita (Okayama University)

1. Introduction

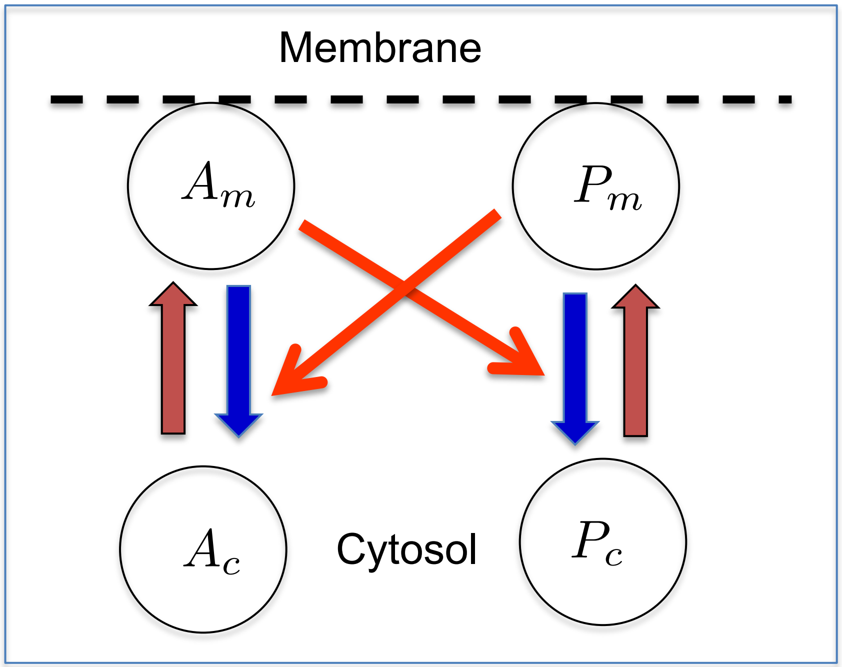
In the asymmetric cell division of *C. elegans* embryo, anterior PAR protein (aPAR) and posterior PAR protein (pPAR) are exclusively formed in an asymmetrical manner, and in the maintenance phase a segregation pattern is created for the polarizations of aPAR and pPAR.

PAR polarity of C. *Elegans* embryo cell



We remark that PAR proteins are mostly upstream regulators that control the downstream proteins and a series of the processes of asymmetric cell division.

A model for asymmetric cell division (by Seirin-Lee – Shibata 2015)



$$\partial_t P_m = D_m \nabla^2 P_m - F_{off} P_m + F_{on} P_c$$

$$\partial_t P_c = D_c \nabla^2 P_c + F_{off} P_m - F_{on} P_c$$

$$\partial_t A_m = \overline{D}_m \nabla^2 A_m - \overline{F}_{off} A_m + \overline{F}_{on} A_c$$

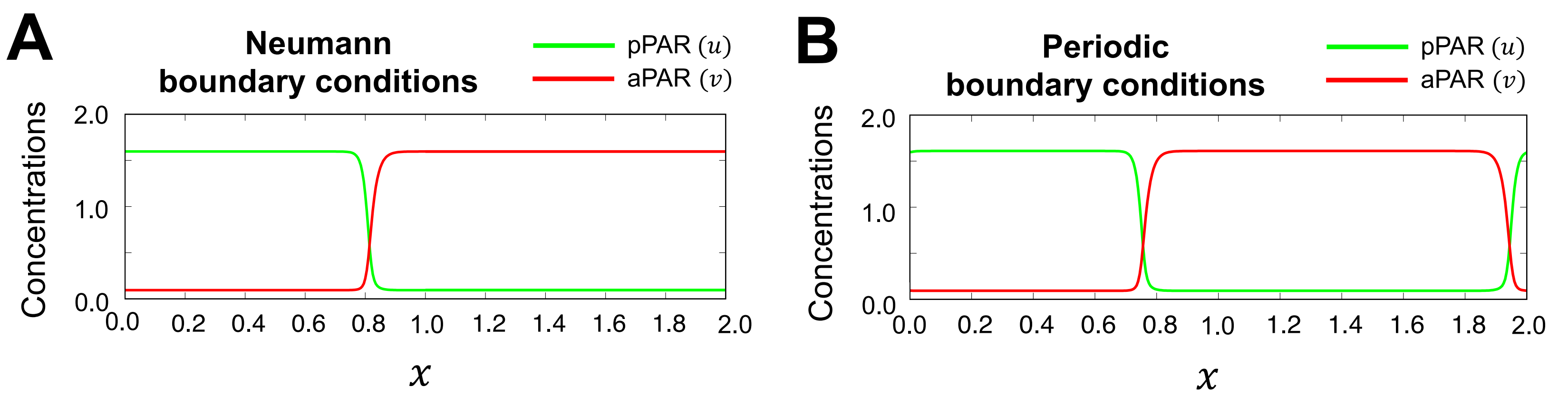
$$\partial_t A_c = \overline{D}_c \nabla^2 A_c + \overline{F}_{off} A_m - \overline{F}_{on} A_c$$

Those off-rate functions depend on A_m and P_m respectively.

$$F_{off} = \alpha + \frac{K_1 A_m^2}{K + A_m^2}, \qquad \overline{F}_{off} = \overline{\alpha} + \frac{\overline{K_1} P_m^2}{\overline{K} + P_m^2},$$

$$F_{on} = \gamma, \qquad \overline{F}_{on} = \overline{\gamma},$$

Numerical simulations for the model demonstrating the segregation pattern



In Seirin-Lee – Shibata 2015 they also consider a 2-component system which is formally reduced from the 4-component system.

Although the two-component system is interesting by itself, the reduction seems to be not so mathematically reasonable.

We take a different approach to the 4-component system (M-Seirin-Lee 2021).

Assuming

$$F_{\text{off}}(A_m) = \alpha + \frac{K_1 {A_m}^2}{K + {A_m}^2} = \alpha + \frac{K_1}{K} A_m^2 + O(A_m^4) \approx \alpha + \frac{K_1}{K} A_m^2,$$

$$\overline{F}_{\text{off}}(P_m) = \overline{\alpha} + \frac{\overline{K_1} {P_m}^2}{\overline{K} + {P_m}^2} = \overline{\alpha} + \frac{\overline{K_1}}{\overline{K}} {P_m}^2 + O(P_m^4) \approx \overline{\alpha} + \frac{\overline{K_1}}{\overline{K}} {P_m}^2$$

By putting

$$k := K_1/K \quad , \qquad \tau := (\overline{K}/\overline{K}_1)k,$$

and

$$u_1 := P_m, \quad v_1 := P_c, \quad u_2 := A_m, \quad v_2 := A_c, \quad d_1 := D_m, \quad D_1 := D_c,$$

$$d_2 := \tau \overline{D}_m, \quad D_2 := \tau \overline{D}_c, \quad \gamma_1 := \gamma, \quad \gamma_2 := \tau \overline{\gamma}, \quad \alpha_1 = \alpha, \quad \alpha_2 := \tau \overline{\alpha}$$

we consider the modified system:

(4system)

$$
\begin{aligned}
\partial_t u_1 &= d_1 \Delta u_1 - (\alpha_1 + k u_2^2) u_1 + \gamma_1 v_1, \\
\partial_t v_1 &= D_1 \Delta v_1 + (\alpha_1 + k u_2^2) u_1 - \gamma_1 v_1, \\
\tau \partial_t u_2 &= d_2 \Delta u_2 - (\alpha_2 + k u_1^2) u_2 + \gamma_2 v_2, \\
\tau \partial_t v_2 &= D_2 \Delta v_2 + (\alpha_2 + k u_1^2) u_2 - \gamma_2 v_2,
\end{aligned}
$$

We deal with this system in a bounded domain $\Omega \subset \mathbb{R}^N$ with the Neumann boundary condition and initial data:

$$
\begin{cases} u_1(x,0) = u_{1,0}(x) \geq 0, \quad u_2(x,0) = u_{2,0}(x) \geq 0, \\ v_1(x,0) = v_{1,0}(x) \geq 0, \quad v_2(x,0) = v_{2,0}(x) \geq 0 \end{cases} \qquad (x \in \overline{\Omega}),
$$

$u_{i,0}$ $v_{i,0} (i = 1, 2)$ are L^∞ and not identically zero.

2. Basic results for the model equations

Standing assumption: $$d_1 < D_1 \quad \text{and} \quad d_2 < D_2$$

Lemma (M--Seirin-Lee)

The system has a unique classical solution $(u_1(x,t), u_2(x,t), v_1(x,t), v_2(x,t))$ satisfying

$$u_1(x,t), u_2(x,t), v_1(x,t), v_2(x,t) > 0 \ (x \in \overline{\Omega})$$

and it exists time global if $1 \leq N \leq 3$.

The system has the property:

Mass conservation:

$$\int_\Omega (u_1(x,t) + v_1(x,t))dx = \text{constant}, \qquad \int_\Omega (u_2(x,t) + v_2(x,t))dx = \text{constant}.$$

because of

$$\frac{d}{dt}\int_\Omega (u_1(x,t) + v_1(x,t))dx = 0, \qquad \frac{d}{dt}\int_\Omega (u_2(x,t) + v_2(x,t))dx = 0.$$

Put

$$m_1 := \langle u_1 \rangle + \langle v_1 \rangle, \qquad m_2 := \langle u_2 \rangle + \langle v_2 \rangle,$$

$$\langle \cdot \rangle := \frac{1}{|\Omega|} \int_\Omega \cdot \, dx$$

By the change of variables

$$z_1 = (d_1/D_1)u_1 + v_1, \qquad z_2 = (d_2/D_2)u_2 + v_2$$

we convert the system to

$$\partial_t u_1 = d_1 \Delta u_1 - (\alpha_1 + k u_2^2) u_1 - (\gamma_1 d_1/D_1) u_1 + \gamma_1 z_1,$$

$$(1 - d_1/D_1)\partial_t u_1 + \partial_t z_1 = D_1 \Delta z_1,$$

$$\tau \partial_t u_2 = d_2 \Delta u_2 - (\alpha_2 + k u_1^2) u_2 - (\gamma_1 d_2/D_2) u_2 + \gamma_2 z_2,$$

$$\tau (1 - d_2/D_2)\partial_t u_2 + \tau \partial_t z_2 = D_2 \Delta z_2.$$

Lyapunov function

$$\mathcal{E}(\boldsymbol{u},\boldsymbol{z}) := \int_\Omega \Big[\frac{d_1}{2}|\nabla u_1|^2 + \frac{d_2}{2}|\nabla u_2|^2 + \frac{\alpha_1+\gamma_1 d_1/D_1}{2}u_1^2$$

$$+\frac{\alpha_2+\gamma_2 d_2/D_2}{2}u_2^2 + \frac{k}{2}u_1^2u_2^2 + \frac{\theta_1}{2}z_1^2 + \frac{\theta_2}{2}z_2^2\Big]\,dx,$$

where $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{z} = (z_1, z_2)$. $\theta_i := \dfrac{\gamma_i}{1-d_i/D_i}$ $(i=1,2)$

Indeed, we can check

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{u}(\cdot,t),\boldsymbol{z}(\cdot,t))$$

$$= -\int_\Omega [(\partial_t u_1)^2 + \tau(\partial_t u_2)^2 + \theta_1 D_1|\nabla z_1|^2 + (\theta_2/\tau)D_2|\nabla z_2|^2]dx \le 0$$

3. Stationary problem

$$d_1 \Delta u_1 - (\alpha_1 + k u_2^2) u_1 - (\gamma_1 d_1 / D_1) u_1 + \gamma_1 z_1 = 0, \qquad \Delta z_1 = 0,$$
$$d_2 \Delta u_2 - (\alpha_2 + k u_1^2) u_2 - (\gamma_2 d_2 / D_2) u_2 + \gamma_2 z_2 = 0, \qquad \Delta z_2 = 0,$$

with

$$m_i = (1 - d_i / D_i) \langle u_i \rangle + \langle z_i \rangle \qquad (i = 1, 2).$$

These equations turn to be

$$\text{(SE)} \quad \begin{aligned} d_1 \Delta u_1 - (\beta_1 + k u_2^2) u_1 + \gamma_1 \{ m_1 - (1 - d_1 / D_1) \langle u_1 \rangle \} = 0, \\ d_2 \Delta u_2 - (\beta_2 + k u_1^2) u_2 + \gamma_2 \{ m_2 - (1 - d_2 / D_2) \langle u_2 \rangle \} = 0. \end{aligned}$$

where

$$\beta_i := \alpha_i + \gamma_i d_i / D_i \qquad (i = 1, 2)$$

We note that the corresponding to a solution (u_1^*, u_2^*) to (SE)

$$(u_1^*, z_1^*, u_2^*, z_2^*) = (u_1^*,\ m_1 - (1 - d_1/D_1)\langle u_1^* \rangle,\ u_2^*,\ m_2 - (1 - d_2/D_2)\langle u_2^* \rangle)$$

gives an equilibrium solution to (4system).

Variational characterization

Define

$$\mathcal{E}_s(\boldsymbol{u}) := \int_\Omega \left\{ \frac{d_1}{2}|\nabla u_1|^2 + \frac{d_2}{2}|\nabla u_2|^2 + \frac{\beta_1}{2}u_1^2 + \frac{\beta_2}{2}u_2^2 + \frac{k}{2}u_1^2 u_2^2 \right\} dx$$
$$+ \frac{\gamma_1|\Omega|}{2(1-d_1/D_1)}\left\{m_1 - (1-d_1/D_1)\langle u_1\rangle\right\}^2$$
$$+ \frac{\gamma_2|\Omega|}{2(1-d_2/D_2)}\left\{m_2 - (1-d_2/D_2)\langle u_2\rangle\right\}^2 .$$

It is easy to see that (SE) with the Neumann boundary condition is the Euler-Lagrange equation of $\mathcal{E}_s$.

(Q) A local minimizer is stable in the 4-component system?

Lemma (stability) M--Serin-Lee

Let $\boldsymbol{u}^*$ be a local minimizer of $\mathcal{E}_s$ and let $\boldsymbol{z}^* = (z_1^*, z_2^*)$ be defined as

$$z_i^* := m_i - (1 - d_i/D_i)\langle u_i^* \rangle \quad (i = 1, 2).$$

Then given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|(\boldsymbol{u}(\cdot, 0), \boldsymbol{z}(\cdot, 0)) - (\boldsymbol{u}^*, \boldsymbol{z}^*)\|_{H^1} < \delta$$

implies

$$\|(\boldsymbol{u}(\cdot, t), \boldsymbol{z}(\cdot, t)) - (\boldsymbol{u}^*, \boldsymbol{z}^*)\|_{H^1} < \tilde{C}\varepsilon \quad (t \geq 0),$$

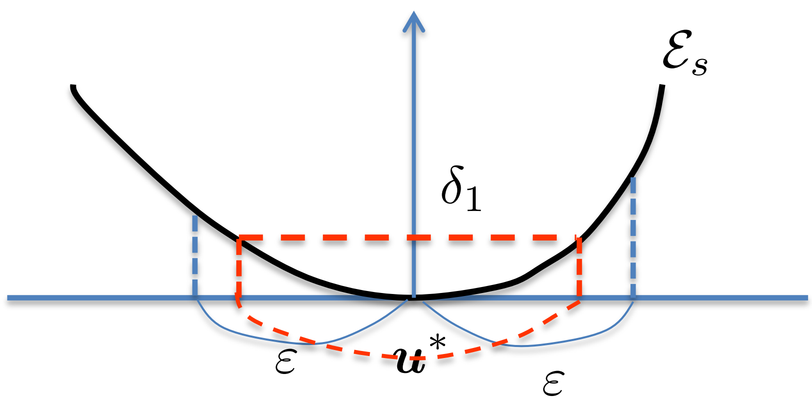
for a constant $\tilde{C} > 0$.

In the view of Lemma (stability), in order to prove the existence of stable nonconstant equilibrium solutions of (4system), it suffices to show the existence of a nonconstant minimizer of $\mathcal{E}_s$.

Key lemma (Latos-Suzuki)

Let $\boldsymbol{u}^* = (u_1^*, u_2^*)$ be a local minimizer of $\mathcal{E}_s(\boldsymbol{u})$ $(\boldsymbol{u} \in H^1(\Omega)^2)$.
Then there exists an $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1/4]$ we can take $\delta_1 = \delta_1(\varepsilon) > 0$ so that $\|\boldsymbol{u} - \boldsymbol{u}^*\|_{H^1} < \varepsilon$ holds if

$$\mathcal{E}_s(\boldsymbol{u}) - \mathcal{E}_s(\boldsymbol{u}^*) < \delta_1 \quad \text{with} \quad \|\boldsymbol{u} - \boldsymbol{u}^*\|_{H^1} < \varepsilon_1.$$



Lemma (stability) is proved by using this lemma.

4. Spectral comparison

Around an equilibrium solution $(u_1^*, v_1^*, u_2^*, v_2^*)$, we consider the linearized operator

$$\boldsymbol{\mathcal{L}}\begin{pmatrix}\phi_1\\ \psi_1\\ \phi_2\\ \psi_2\end{pmatrix} := \begin{pmatrix} -d_1\Delta\phi_1 + (\alpha_1 + k(u_2^*)^2)\phi_1 + 2k(u_1^*u_2^*)\phi_2 - \gamma_1\psi_1 \\ -D_1\Delta\psi_1 - (\alpha_1 + k(u_2^*)^2)\phi_1 - 2k(u_1^*u_2^*)\phi_2 + \gamma_1\psi_1 \\ -d_2\Delta\phi_2 + (\alpha_1 + k(u_1^*)^2)\phi_2 + 2k(u_1^*u_2^*)\phi_1 - \gamma_2\psi_2 \\ -D_2\Delta\phi_2 - (\alpha_1 + k(u_1^*)^2)\phi_2 - 2k(u_1^*u_2^*)\phi_1 + \gamma_2\psi_2 \end{pmatrix}$$

$$\begin{aligned}\mathrm{Dom}(\boldsymbol{\mathcal{L}}) =& \{(\phi_1, \psi_1, \phi_2, \psi_2)^T \in H^2(\Omega;\mathbb{R}^4) : \\ & \partial\phi_i/\partial\boldsymbol{n} = \partial\psi_i/\partial\boldsymbol{n} = 0 \ \text{ on } \ \partial\Omega, \quad \langle\phi_i\rangle + \langle\psi_i\rangle = 0 \quad (i = 1, 2)\}\end{aligned}$$

$$\mathcal{L}_1\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} := \begin{pmatrix} -d_1\Delta\varphi_1 + (\beta_1 + k(u_2^*)^2)\varphi_1 + 2k(u_1^*u_2^*)\varphi_2 + \gamma_1(1 - d_1/D_1)\langle\varphi_1\rangle \\ -d_2\Delta\varphi_2 + (\beta_1 + k(u_1^*)^2)\varphi_2 + 2k(u_1^*u_2^*)\varphi_1 + \gamma_2(1 - d_2/D_2)\langle\varphi_2\rangle \end{pmatrix}$$

$$\begin{aligned}\mathrm{Dom}(\mathcal{L}_1) =& \{(\phi_1, \phi_2)^T \in H^2(\Omega;\mathbb{R}^2) : \\ & \partial\phi_i/\partial\boldsymbol{n} = 0 \ \text{ on } \ \partial\Omega \quad (i = 1, 2)\}\end{aligned}$$

We let $\{\lambda_j\}_{j=1,2,\ldots}$ and $\{\nu_j\}_{j=1,2,\ldots}$ be sets of eigenvalues of $\mathcal{L}$ and $\mathcal{L}_1$ respectively.

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \cdots$$

$$\nu_1 \leq \nu_2 \leq \cdots \leq \nu_j \leq \nu_{j+1} \leq \cdots$$

We have the following spectral comparison result:

Theorem (M-Oshita)

If $\lambda_j \neq 0$ or $\nu_j \neq 0$, then $\lambda_j \nu_j > 0,$ $|\lambda_j| < |\nu_j|$ holds.

Moreover, if $\lambda_j = 0$, then $\nu_j = 0$ holds, and vice versa.

This implies that the numbers of unstable eigenvalues to $\mathcal{L}$ and $\mathcal{L}_1$ coincide.

Proof can be done by modifying the arguments in M (2012).

(cf. Bates-Fife (1990), Ohinishi-Nishiura (1998))

4. Profile of stable solutions

We first introduce previous results.

<u>Lemma (minimizer of the reduced problem)</u>

There is a minimizer $\boldsymbol{u}^* = (u_1^*, u_2^*)$ of $\mathcal{E}_s$ satisfying

$$u_i^*(x) > 0 \quad (x \in \overline{\Omega}), \qquad i = 1, 2.$$

<u>Theorem (M-S.-Lee)</u>

Let $\Omega \subset \mathbb{R}^N$ $(1 \leq N \leq 3)$ be a cylindrical domain as $\Omega = \{x = (x_1, x') \in (0, L) \times D\}$, where D is a bounded domain of $\mathbb{R}^{N-1}$ with smooth boundary. For the diffusion coefficients assume $d_i < D_i$ $(i = 1, 2)$. Then there are positive numbers $\overline{\alpha}$, $\overline{d}$ and $\overline{r}$ such that for

$$\alpha_i \leq \overline{\alpha}, \quad d_i \leq \overline{d}, \quad d_i/\alpha_i \leq \overline{r} \quad (i = 1, 2),$$

the system (4system) in Ω with N.B.C possesses a stable nonconstant equilibrium solution.

Sketch of Proof: For simplicity assume $\Omega = (0, L)$.

Put
$$\omega_i := \sqrt{d_i/\beta_i} \qquad (i = 1, 2)$$

Assume
$$d_i \ll D_i, \quad \beta_i \ll 1 \quad (i = 1, 2), \qquad \ell > \omega_1, \quad L - \ell > \omega_2.$$

$$(\beta_i = \alpha_i + \gamma_i d_i / D_i, \quad \text{so} \quad \alpha_i \approx \beta_i)$$

Define

$$U_1(x; \ell) := \mu_1(\ell) \left(1 - \frac{\cosh(x/\omega_1)}{\cosh(\ell/\omega_1)}\right),$$

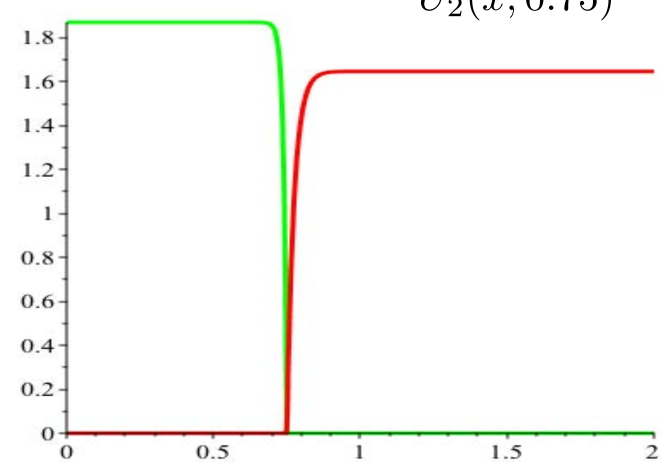
$$U_2(x; \ell) := \mu_2(\ell) \left(1 - \frac{\cosh((L - x)/\omega_2)}{\cosh((L - \ell)/\omega_2)}\right)$$

$$\mu_1(\ell) := \frac{m_1}{\frac{1 - d_1/D_1}{L}\{\ell - \omega_1 \tanh(\ell/\omega_1)\} + \beta_1/\gamma_1},$$

$$\mu_2(\ell) := \frac{m_2}{\frac{1 - d_2/D_2}{L}\{L - \ell - \omega_2 \tanh((L - \ell)/\omega_2)\} + \beta_2/\gamma_2},$$

$U_1(x; 0.75)$

$U_2(x; 0.75)$



0.75

Define
$$\tilde{U}_1(x) := \begin{cases} U_1(x;\ell) & (0 \le x \le \ell), \\ 0 & (\ell \le x \le L), \end{cases}$$

$$\tilde{U}_2(x) := \begin{cases} 0 & (0 \le x \le \ell), \\ U_2(x;\ell) & (\ell \le x \le L). \end{cases}$$

Then

$$d_1(\tilde{U}_1)_{xx} - (\beta_1 + k(\tilde{U}_2)^2)\tilde{U}_1 + \gamma_1\{m_1 - (1 - d_1/D_1)\langle \tilde{U}_1 \rangle\} = \begin{cases} 0 & (0 < x < \ell), \\ \gamma_1\beta_1 & (\ell < x < L), \end{cases}$$

$$d_2(\tilde{U}_2)_{xx} - (\beta_2 + k(\tilde{U}_1)^2)\tilde{U}_2 + \gamma_2\{m_2 - (1 - d_2/D_2)\langle \tilde{U}_2 \rangle\} = \begin{cases} \gamma_2\beta_2 & (0 < x < \ell), \\ 0 & (\ell < x < L). \end{cases}$$

and

$$\mathcal{E}_s(\tilde{U}_1, \tilde{U}_2) = \int_0^\ell \left(\frac{1}{2}((U_1)_x)^2 + \frac{\beta_1}{2} U_1^2 \right) dx + \int_\ell^L \left(\frac{1}{2}((U_2)_x)^2 + \frac{\beta_2}{2} U_2^2 \right) dx$$
$$+ \frac{\gamma_1 L}{2(1 - d_1/D_1)} \left(m_1 - \frac{1 - d_1/D_1}{L} \int_0^\ell U_1(x) dx \right)^2 + \frac{\gamma_2 L}{2(1 - d_2/D_2)} \left(m_2 - \frac{1 - d_2/D_2}{L} \int_\ell^L U_2(x) dx \right)^2$$

If

$$\alpha_i \ll 1, \quad d_i \ll 1, \quad d_i/\alpha_i \ll 1 \quad (i=1,2),$$

(in the sequel $\beta_i \ll 1 \quad (i=1,2)$),

then for any constant solution $(\overline{u}_1, \overline{u}_2)$

$$\mathcal{E}_s(\tilde{U_1}, \tilde{U_2}) < \mathcal{E}_s(\overline{u}_1, \overline{u}_2)$$

holds.

This implies the existence of a stable nonconstant equilibrium solution.

In order to estimate the energy clearly, take the following scaling:

(S) $\quad \alpha_i = \varepsilon \tilde{\alpha}_i, \qquad d_i = \varepsilon^{1+\delta} \tilde{d}_i, \quad 0 < \delta \leq 1 \qquad (i=1,2),$

Then

$$\beta_i = \varepsilon \tilde{\beta}_i(\varepsilon), \qquad \tilde{\beta}_i(\varepsilon) := \tilde{\alpha}_i + \varepsilon^{\delta} \tilde{d}_i \gamma_i / D_i \quad (i=1,2)$$

The equations are written as

$$\varepsilon^{\delta}\tilde{d}_1(u_1)_{xx} - (\tilde{\beta}_1(\varepsilon) + (k/\varepsilon)u_2^2)u_1 + (\gamma_1/\varepsilon)\{m_1 - (1 - \varepsilon^{1+\delta}\tilde{d}_1/D_1)\langle u_1\rangle\} = 0,$$

$$\varepsilon^{\delta}\tilde{d}_2(u_2)_{xx} - (\tilde{\beta}_1(\varepsilon) + (k/\varepsilon)u_1^2)u_2 + (\gamma_2/\varepsilon)\{m_2 - (1 - \varepsilon^{1+\delta}\tilde{d}_2/D_2)\langle u_2\rangle\} = 0.$$

and

$$\begin{aligned}\frac{1}{\varepsilon}\mathcal{E}_s(\boldsymbol{u}) := \int_{\Omega} &\left\{ \frac{\varepsilon^{\delta}\tilde{d}_1}{2}|(u_1)_x|^2 + \frac{\varepsilon^{\delta}\tilde{d}_2}{2}|(u_2)_x|^2 + \frac{\tilde{\beta}_1(\varepsilon)}{2}u_1^2 + \frac{\tilde{\beta}_2(\varepsilon)}{2}u_2^2 + \frac{k}{2\varepsilon}u_1^2u_2^2 \right\} dx \\ &+ \frac{\gamma_1 L}{2\varepsilon(1 - \varepsilon^{1+\delta}\tilde{d}_1/D_1)}\left\{m_1 - (1 - \varepsilon^{1+\delta}\tilde{d}/D_1)\langle u_1\rangle\right\}^2 \\ &+ \frac{\gamma_2 L}{2\varepsilon(1 - \varepsilon^{1+\delta}\tilde{d}_2/D_2)}\left\{m_2 - (1 - \varepsilon^{1+\delta}\tilde{d}_2/D_2)\langle u_2\rangle\right\}^2.\end{aligned}$$

In view of

$$\mu_1(\ell) = \frac{Lm_1}{\ell} + O(\varepsilon^{\delta/2}), \qquad \mu_2(\ell) = \frac{Lm_2}{L-\ell} + O(\varepsilon^{\delta/2})$$

we have

$$\frac{1}{\varepsilon}\mathcal{E}_s(\tilde{U}_1, \tilde{U}_2) = \frac{1}{2}\frac{(Lm_1)^2\tilde{\alpha}_1}{\ell} + \frac{1}{2}\frac{(Lm_2)^2\tilde{\alpha}_2}{L-\ell} + O(\varepsilon^{\delta/2})$$

$$\geq \frac{L}{2}(m_1\sqrt{\tilde{\alpha}_1} + m_2\sqrt{\tilde{\alpha}_2})^2 + O(\varepsilon^{\delta/2})$$

$$\text{for} \quad \ell = \ell^* := \frac{Lm_1\sqrt{\tilde{\alpha}_1}}{m_1\sqrt{\tilde{\alpha}_1} + m_2\sqrt{\tilde{\alpha}_2}}$$

Since $\frac{1}{\varepsilon}\mathcal{E}_s(u_1^\varepsilon, u_2^\varepsilon) \le \frac{1}{\varepsilon}\mathcal{E}_s(\tilde{U}_1, \tilde{U}_2)$ for the minimizer $(u_1^\varepsilon, u_2^\varepsilon)$

we have an upper estimate

$$\frac{1}{\varepsilon}\mathcal{E}_s(u_1^\varepsilon, u_2^\varepsilon) \le \frac{L}{2}(m_1\sqrt{\tilde{\alpha}_1} + m_2\sqrt{\tilde{\alpha}_2})^2 + O(\varepsilon^{\delta/2})$$

(M– Seirin-Lee)

Moreover, we obtain

Lemma (M-Oshita)

Let $\boldsymbol{u}_\varepsilon^* = (u_{1\varepsilon}^*, u_{2\varepsilon}^*)$ be the minimizer of $\mathcal{E}_s$ with (S). Then

$$\frac{L}{2}(\sqrt{\tilde{\alpha}_1}m_1 + \sqrt{\tilde{\alpha}_2}m_2)^2 - \rho_1(\varepsilon) \le \frac{1}{\varepsilon}\mathcal{E}_s(\boldsymbol{u}_\varepsilon^*) \le \frac{L}{2}(\sqrt{\tilde{\alpha}_1}m_1 + \sqrt{\tilde{\alpha}_2}m_2)^2 + \rho_2(\varepsilon),$$

where $\rho_1(\varepsilon) = O(\varepsilon^{1/2})$ and $\rho_2(\varepsilon) = O(\varepsilon^{\delta/2})$.

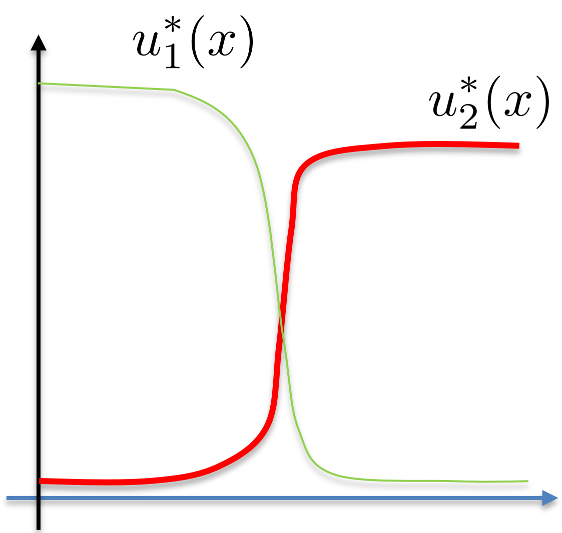
Theorem (M-Oshita)

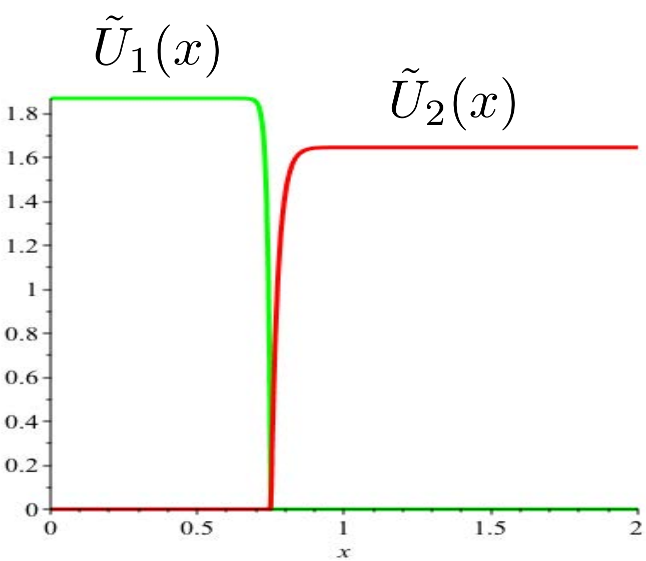
There exists an equilibrium solution $\boldsymbol{u}^* = (u_1^*, u_2^*)$ satisfying

$$(u_1^*)_x (u_2^*)_x < 0 \qquad (0 < x < L)$$

and

$$\frac{L}{2}(\sqrt{\tilde{\alpha}_1} m_1 + \sqrt{\tilde{\alpha}_2} m_2)^2 - \rho_1(\varepsilon) \le \frac{1}{\varepsilon}\mathcal{E}_s(\boldsymbol{u}^*) \le \frac{L}{2}(\sqrt{\tilde{\alpha}_1} m_1 + \sqrt{\tilde{\alpha}_2} m_2)^2 + \rho_2(\varepsilon)$$





(Future problems)

(i) A rigorous proof to show that the equilibrium solutions with monotone profile is a minimizer.

(ii) The existence of a stable/unstable solution with multi layers;

Although we can prove that solutions obtained by reflection of the monotone one is unstable, it remains to verify if there exists a stable solution with multi-layers or not.

(iii) The existence of a stable solution with a transition layer in a higher-dimensional domain;

(iv) Different parameter regime allowing the segregation pattern should be examined.

(v) Free boundary problem in the singular limit $\varepsilon \to 0$.

Thank you for sharing your time!

In memory of Professor Masayasu Mimura for his great achievements in the theory of pattern formation arising in reaction-diffusion systems

"Speed-up of traveling waves by negative chemotaxis"

Quentin Griette (Université Le Havre Normandie, France)

We consider the traveling wave speed for Fisher-KPP (FKPP) fronts under the influence of chemotaxis and provide an almost complete picture of its asymptotic dependence on parameters representing the strength and length-scale of chemotaxis.

Our study is based on the convergence to the porous medium FKPP traveling wave and a hyperbolic FKPP-Keller-Segel traveling wave in certain asymptotic regimes. In this way, it clarifies the relationship between three equations that have each garnered intense interest on their own. Our proofs involve a variety of techniques ranging from entropy methods and decay of oscillations estimates to a general description of the qualitative behavior to the hyperbolic FKPP-Keller-Segel equation. For this latter equation, we, as a part of our limiting arguments, establish an explicit lower bound on the minimal traveling wave speed and provide a new construction of traveling waves that extends the known existence range to all parameter values.

This is a joint work with Chris Henderson and Olga Turanova.

Traveling waves in repulsive Keller-Segel models

Quentin Griette (Université Le Havre Normandie)

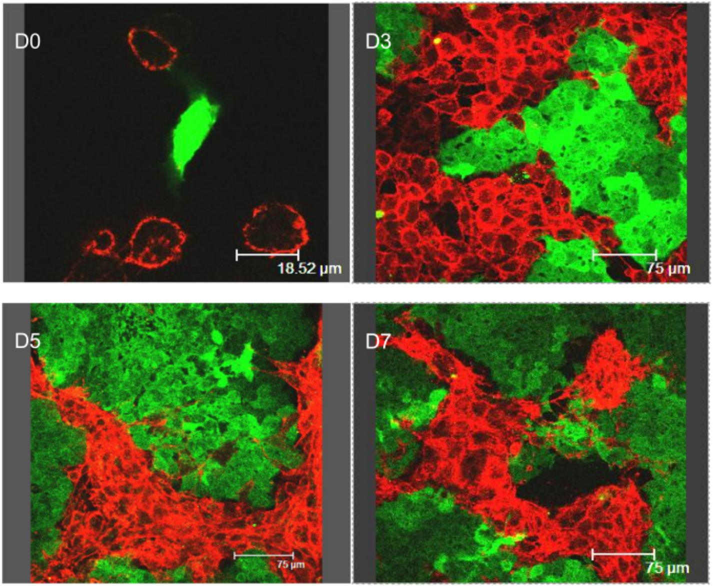
quentin.griette@univ-lehavre.fr

ICMMA2023: "Reaction-diffusion systems: from the past to the future"
in memory of Prof. Masayasu Mimura

November 2nd, 2023

PART I:
The hyperbolic-elliptic model

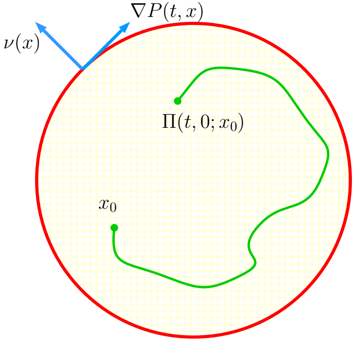
joint works with Xiaoming Fu, Pierre Magal, Min Zhao.



[1] Jennifer Pasquier et al. "Different Modalities of Intercellular Membrane Exchanges Mediate Cell-to-cell P-glycoprotein Transfers in MCF-7 Breast Cancer Cells". *J. Biol. Chem.* 287.10 (2012), pp. 7374–7387. DOI: 10.1074/jbc.M111.312157.

The cell-cell repulsion model

$$\begin{cases} \partial_t u - \chi \nabla \cdot \big(u \nabla P\big) = u\big(1-u\big), \\ P - \sigma^2 \Delta P = u, \\ \nu \cdot \nabla P = 0, \end{cases}$$



[2] Xiaoming Fu, Quentin Griette, and Pierre Magal. "A cell-cell repulsion model on a hyperbolic Keller-Segel equation". *J. Math. Biol.* 80.7 (2020), pp. 2257–2300. DOI: 10.1007/s00285-020-01495-w.

Related models

- The Patlak-Keller-Segel equation modeling chemotaxis (Patlak 1953, Keller and Segel 1970)

$$\begin{cases} u_t = \kappa \Delta u - \chi \nabla \cdot (u \nabla c) \\ \varepsilon c_t = \eta \Delta c + \beta n - \alpha c, \end{cases}$$

  see also Calvez and Corrias 2008, Desvillettes et al 2019.

- The porous medium equation with KPP source

$$u_t = \nabla \cdot (u \nabla u) + u(1-u)$$

  see de Pablo and Vazquez 1991, Vazquez 2007.

- in 2006, Armstrong, Painter and Sherratt proposed a model for cell-cell adhesion modeled by a nonlocal gradient.

$$u_t = u_{xx} - (uK(u))_x,$$

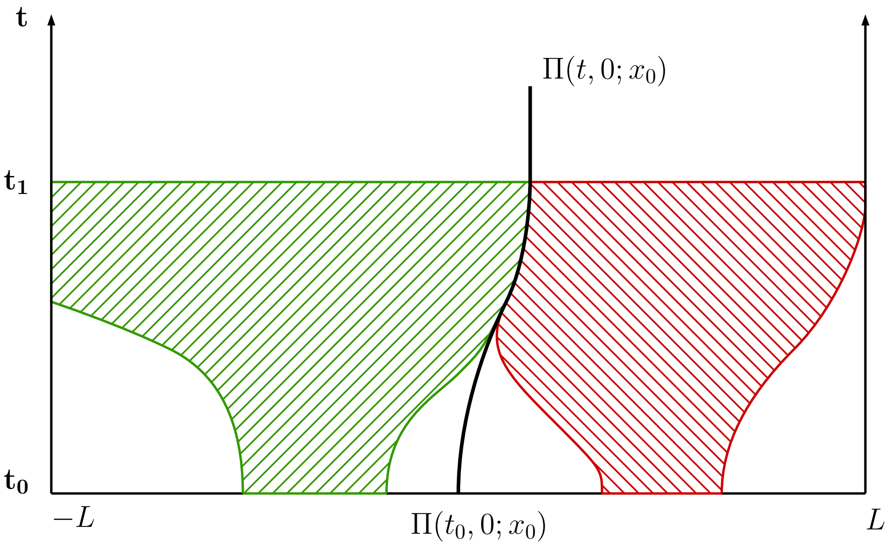
  where $K(u) = \alpha \int_{-1}^{1} g(u(x+x_0))\omega(x_0)\mathrm{d}x_0$.

- A full model was proposed by Ducrot et al (2011) with a porous medium equation with contact inhibition.

Two-species model

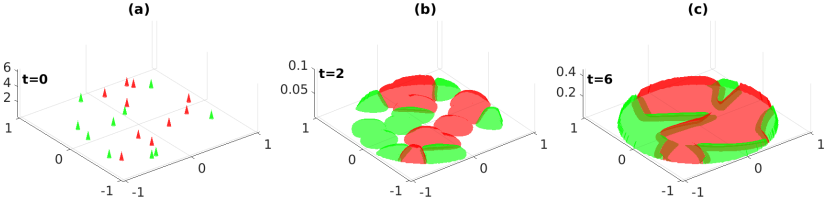
$$\begin{cases} \partial_t u_1 - \chi_1 \nabla \cdot \big(u_1 \nabla P \big) = u_1 \big(r_1 - a_{11} u_1 - a_{12} u_2 \big), \\ \partial_t u_2 - \chi_2 \nabla \cdot \big(u_2 \nabla P \big) = u_2 \big(r_2 - a_{21} u_1 - a_{22} u_2 \big), \\ P - \sigma^2 \Delta P = u_1 + u_2, \\ \nu \cdot \nabla P = 0, \end{cases}$$

Preservation of segregation when $\chi_1 = \chi_2$

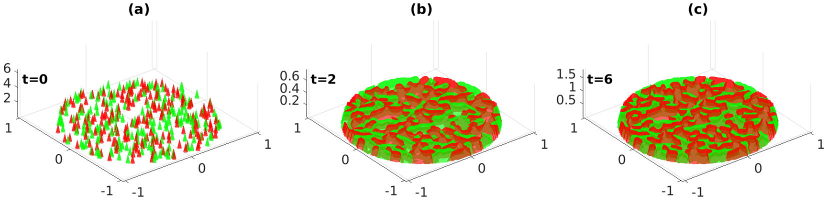


Monolayer cell experiment in the Petri dish

sparsely seeded initial condition

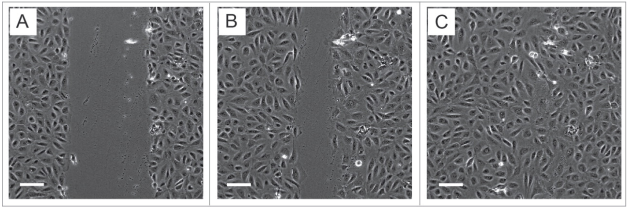


densely seeded initial condition



The one-dimensional model: motivation

$$\begin{cases} \partial_t u(t,x) - \chi \partial_x (u(t,x) \partial_x p(t,x)) = u(1-u), & x \in \mathbb{R}, t > 0 \\ p(t,x) - \sigma^2 \partial_{xx} p(t,x) = u(t,x), & x \in \mathbb{R}, t > 0 \\ u(t=0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$



Wound healing experiments

[3] James E. N. Jonkman et al. "An introduction to the wound healing assay using live-cell microscopy". *Cell Adhesion & Migration* 8.5 (2014). PMID: 25482647, pp. 440–451. DOI: 10.4161/cam.36224.

The one-dimensional one-species model

We focus on the one-dimensional model

$$\begin{cases} \partial_t u(t,x) - \chi \partial_x (u(t,x)\partial_x p(t,x)) = u(1-u), & x \in \mathbb{R}, t > 0 \\ u(t=0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

As before p is determined by the following equation

$$p(t,x) - \sigma^2 \partial_{xx} p(t,x) = u(t,x).$$

Equivalently,

$$p(t,x) = (\rho \star u)(t,x) = \int_{\mathbb{R}} \rho(x) u(t,x-y) \mathrm{d}y, \qquad \rho(x) := \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}}$$

Solution integrated along the characteristics

We transform the original model by using the characteristic curves:

$$\begin{cases} \partial_t u(t,x) - \chi \partial_x u(t,x) \partial_x p(t,x) = u\left(1 + \chi \dfrac{v-u}{\sigma^2} - u\right), \quad x \in \mathbb{R}, t > 0 \\ p(t,x) = (\rho \star u)(t,x) \end{cases}$$

where $\rho(x) := \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}}$, is split in two equations: the equation of the **characteristic curves**

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \Pi(t,x) = -\chi(\rho_x \star u)(t, \Pi(t,x)), \\ \Pi(t=0,x) = x, \end{cases} \tag{1}$$

and the **local dynamics on a characteristic**

$$\frac{\mathrm{d}}{\mathrm{d}t} \hat{u}(t,x) = \hat{u}(t,x)\big(1 + \hat{\chi}(\rho \star u)(t, \Pi(t,x)) - (1+\hat{\chi})\, \hat{u}(t,x)\big), \tag{2}$$

where $\hat{u}(t,x) = h\big(t, \Pi(t,x)\big)$ and $\hat{\chi} = \frac{\chi}{\sigma^2}$. (1)–(2) define a SOLUTION INTEGRATED ALONG THE CHARACTERISTICS.

The Cauchy problem

Here L^1_η is the L^1 space for the measure $\frac{\eta}{2}e^{-\eta|x|}\mathrm{d}x$.

Theorem (Solution integrated along the characteristics)

Let $u_0 \in L^\infty(\mathbb{R})$, $u_0(x) \geq 0$. There exists $\tau^*(u_0) \in (0, +\infty]$ such that

1. *For each $\tau \in [0, \tau)$, there is a unique $u(t,x) \in C^0\big([0,\tau], L^1_\eta(\mathbb{R})\big)$ which is a solution integrated along the characteristics and satisfies $u(t=0,x) = u_0(x)$.*
2. *For each $t \geq 0$ we have $u(t,\cdot) \in L^\infty(\mathbb{R})$,*
3. *The map $t \mapsto T_t u_0 := u(t,\cdot)$ is a semigroup which is continuous for the $L^1_\eta(\mathbb{R})$ topology,*
4. *For each $t \geq 0$, the map $u_0 \in L^\infty(\mathbb{R}) \mapsto T_t u_0 = u(t,\cdot)$ is continuous for the L^1_η topology*

In addition: preservation of monotony, continuity and differentiability (as well as superior smoothness) of the initial data.

Discontinuous traveling waves

Theorem (Existence of a sharp traveling wave)

Assume $0 < \hat{\chi} < \bar{\chi}$. There exists a traveling wave traveling at speed $c \in \left(\frac{\sigma\hat{\chi}}{2+\hat{\chi}}, \frac{\sigma\hat{\chi}}{2}\right)$. U satisfies $U(x) = 0$ for all $x \geq 0$, $U(0^-) \geq \frac{2}{2+\hat{\chi}}$. Moreover U is strictly decreasing and differentiable on $(-\infty, 0]$ and a classical solution to

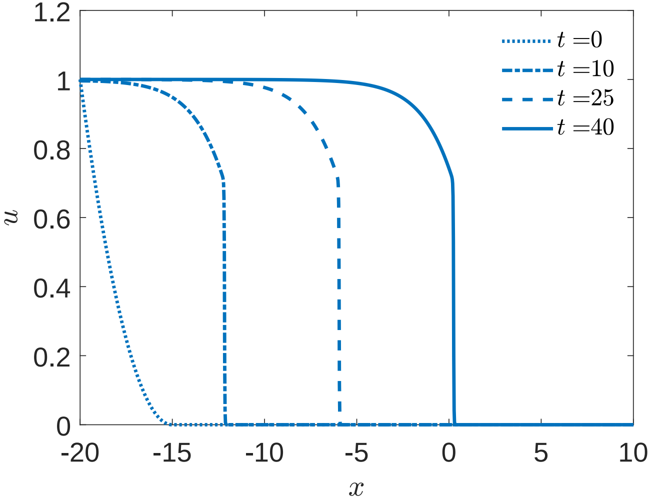
$$-cU' - \chi(UP')' = U(1-U),$$

where $P = (U \star \rho)$.

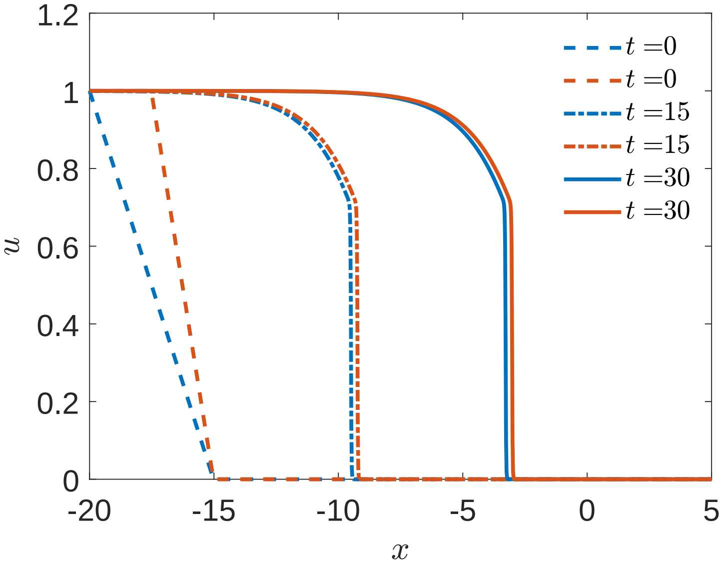
In addition: for solutions of the Cauchy problem with compactly supported initial data, convergence speed of the level sets to the separatric, estimate on the jump size. Non-existence of sharp continuous traveling waves.

[4] Xiaoming Fu, Quentin Griette, and Pierre Magal. "Sharp discontinuous traveling waves in a hyperbolic Keller–Segel equation". *Mathematical Models and Methods in Applied Sciences* (2021). to appear. DOI: 10.1142/S0218202521500214.

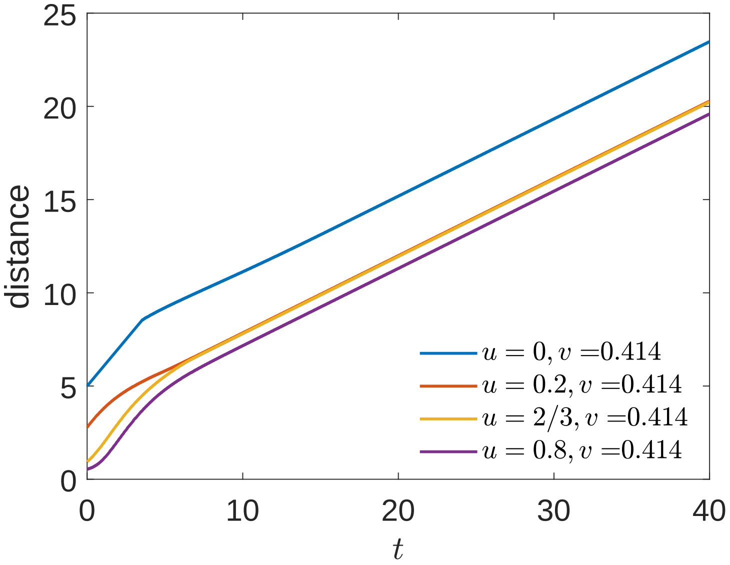
Initial condition $u_0(x) = \frac{(x - x_0)^2}{(L + x_0)^2} \mathbb{1}_{[-L,x_0]}(x)$, $L = 20$, $x_0 = -15$.



Dependency on the initial steepness



Position of the level sets and empirical speed



Formation of a discontinuity

Theorem (Exponential convergence of the level sets)

Let $u_0 \in C^0(\mathbb{R})$, $u_0(x) \geq 0$ *be supported in* $(-\infty, 0)$, *and assume that the behavior of* $u_0(x)$ *near* $x = 0$ *is polynomial:*

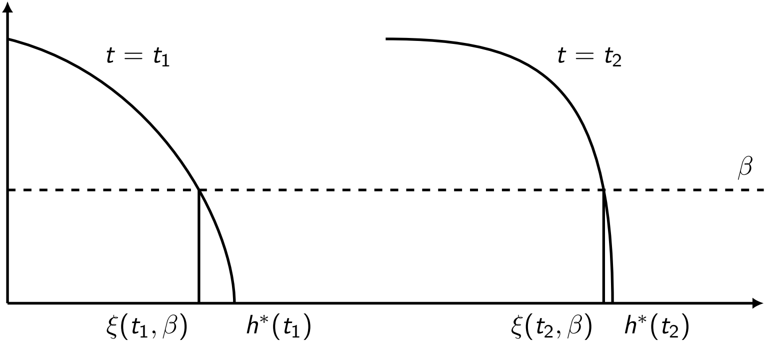
$$u_0(x) \geq \gamma |x|^\alpha,$$

for some $\gamma > 0$ *and* $\alpha \geq 1$. *Let* $\Pi^*(t) := \Pi(t, 0)$ *be the characteristic starting from* $x = 0$ *and* $\xi(t, \beta) := \sup\{x \in \mathbb{R} | u(t, x) = \beta\}$ *be the level set at level* β. *Then for each* $\beta \in (0, \frac{1}{1+\hat{\chi}+\alpha\hat{\chi}})$ *we have*

$$\Pi^*(t) - \left(\frac{\beta}{\gamma}\right)^{\frac{1}{\alpha}} e^{-\frac{\eta}{2\alpha} t} \leq \xi(t, \beta) \leq \Pi^*(t),$$

where $\eta := 1 - \frac{1+\hat{\chi}+\alpha\hat{\chi}}{\beta} \in (0, 1)$.

A cartoon for the formation of the discontinuity



Divergence of characteristics

We quantify the divergence speed of characteristics near the separatrix. We have, for $x < 0$,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Pi(t,0;0)-\Pi(t,0;x)\Big) &\leq -\chi(\rho_x\star u)(t,\Pi(t,0;0)) + \chi(\rho_x\star u)(t,\Pi(t,0;x)) \\
&= \chi\int_{\mathbb{R}}\Big(\rho_x(\Pi(t,0;x)-z)-\rho_x(\Pi(t,0;0)-z)\Big)u(t,z)\mathrm{d}z \\
&= \chi\int_{-\infty}^{\Pi(t,0;x)}\Big(\rho_x(\Pi(t,0;x)-z)-\rho_x(\Pi(t,0;0)-z)\Big)u(t,z)\mathrm{d}z \\
&\quad + \chi\int_{\Pi(t,0;x)}^{\Pi(t,0;0)}\Big(\rho_x(\Pi(t,0;x)-z)-\rho_x(\Pi(t,0;0)-z)\Big)u(t,z)\mathrm{d}z
\end{aligned}
$$

Divergence of characteristics

We quantify the divergence speed of characteristics near the separatrix. We have, for $x < 0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Pi(t,0;0)-\Pi(t,0;x)\Big) &\leq -\chi(\rho_x \star u)(t,\Pi(t,0;0)) + \chi(\rho_x \star u)(t,\Pi(t,0;x)) \\ &= \chi\int_{\mathbb{R}}\Big(\rho_x(\Pi(t,0;x)-z)-\rho_x(\Pi(t,0;0)-z)\Big)u(t,z)\mathrm{d}z \\ &= \chi\int_{-\infty}^{\Pi(t,0;x)}\underbrace{\Big(\rho_x(\Pi(t,0;x)-z)-\rho_x(\Pi(t,0;0)-z)\Big)u(t,z)\mathrm{d}z}_{\leq 0} \\ &\quad + \chi\int_{\Pi(t,0;x)}^{\Pi(t,0;0)}\underbrace{\Big(\rho_x(\Pi(t,0;x)-z)-\rho_x(\Pi(t,0;0)-z)\Big)}_{\leq 1/\sigma^2}u(t,z)\mathrm{d}z\end{aligned}$$

where $\rho(z)=\frac{1}{2\sigma}e^{-\frac{|z|}{\sigma}}$, $\rho_x(z)=-\frac{\mathrm{sign}(z)}{2\sigma^2}e^{-\frac{|z|}{\sigma}}$. Therefore

$$\Pi(t,0;0) > \Pi(t,0;x) \geq \Pi(t,0;0) + xe^{\hat{\chi}\varepsilon t}$$

where $\varepsilon := \sup_{x\leq z\leq 0, s\in[0,t]} u(s,\Pi(s,0;z))$.

On each characteristic, $u(t, \Pi(t, 0; x))$ satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} u(t, \Pi(t, 0; x)) &= u(t, \Pi(t, 0; x)) \Big(1 + \hat{\chi}(\rho \star u)\Big(t, \Pi(t, 0; x)\Big) - (1 + \hat{\chi}) u\Big(t, \Pi(t, 0; x)\Big)\Big) \\ &\geq u(t, \Pi(t, 0; x))\end{aligned}$$

if $u(t, \Pi(t, 0; x))$ is sufficiently small. Therefore

$$u(t, \Pi(t, 0; x)) \geq u_0(x) e^t.$$

Heuristic computation

Assume x is close to $\Pi(t,0;0)$ so that $u(t,x) \leq \varepsilon$.
In the worst case scenario $\Pi(t,0;x) \approx \Pi(t,0;0) + xe^{\hat{\chi}\varepsilon t}$ so that

$$\Pi(0,t;x) \approx -(\Pi(t,0;0) - x)e^{-\hat{\chi}\varepsilon t}.$$

Next by using the dynamics on the characteristics

$$\begin{aligned} u(t,x) = u(t,\Pi(t,0;\Pi(0,t;x))) \geq u_0(\Pi(0,t;x))e^{t} \\ \approx u_0\big(-(\Pi(t,0;0) - x)e^{-\hat{\chi}\varepsilon t}\big)e^{t}, \end{aligned}$$

therefore if $u_0(z)$ is polynomial near 0: $u_0(z) \geq \gamma|z|^{\alpha}$,

$$u(t,x) \gtrapprox \gamma(\Pi(t,0;0) - x)^{\alpha} e^{(1-\alpha\hat{\chi}\varepsilon)t}$$

Preservation of regularity

Proposition

Let $u_0 \in L^\infty(\mathbb{R})$ and $u(t,x)$ be an integrated solution.

1. *if $0 \leq u_0(x) \leq 1$, then $0 \leq u(t,x) \leq 1$. In particular $\tau^*(u_0) = +\infty$.*
2. *if $u_0(x)$ is continuous, then $u \in C^0([0,\tau] \times \mathbb{R})$ for all $\tau < \tau^*(u_0)$.*
3. *if $u_0(x) \in C^1(\mathbb{R})$, then $u \in C^1([0,\tau] \times \mathbb{R})$ for all $\tau < \tau^*(u_0)$. In this case $u(t,x)$ is a classical solution of the hyperbolic problem.*
4. *if $u_0(x)$ is monotone, then $u(t,x)$ has the same monotony for all $0 \leq t < \tau^*(u_0)$.*

The first property comes from an ad hoc argument. Properties 2 - 4 are shown thanks to the following formula

$$u(t,x) = \frac{u_0(\Pi(0,t;x)) \exp\left(\int_0^t 1 + \hat{\chi} p(l, \Pi(l,t;x)) \mathrm{d}l\right)}{1 + (1+\hat{\chi}) u_0(\Pi(0,t;x)) \int_0^t \exp\left(\int_0^l 1 + \hat{\chi} p(\sigma, \Pi(\sigma,t;x)) \mathrm{d}\sigma\right) \mathrm{d}l}$$

which requires the *a priori* definition of Π and p.

Explicit estimate of the jump size

Theorem (Estimate of the jump)

Let $u_0 \in L^\infty(\mathbb{R})$ *be a non-increasing profile supported in* $(-\infty, 0]$ *satisfying*

$$u(-\infty) \leq 1, \qquad u(0^-) > 0.$$

Then:

$$\liminf_{t\to+\infty} u(t, \Pi^*(t)) \geq \frac{2}{2+\hat{\chi}},$$
$$\liminf_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\Pi^*(t) \geq \frac{\sigma\hat{\chi}}{2+\hat{\chi}},$$

where $\Pi^*(t) = \Pi(t, 0)$ *and* $\hat{\chi} = \frac{\chi}{\sigma^2}$*.*

Note that this estimate is better than the one provided by the "convergence of the level sets" theorem. Indeed

$$\frac{2}{2+\hat{\chi}} = \frac{1}{1+\frac{\hat{\chi}}{2}} > \frac{1}{1+\hat{\chi}} \geq \frac{1}{1+\hat{\chi}+\alpha\hat{\chi}}.$$

Non-existence of smooth sharp waves

Our last result is the non-existence of smooth sharp traveling waves. It confirms that discontinuous traveling waves are the "natural" asymptotic shape of compactly supported initial conditions.

Theorem (Non-existence of sharp smooth waves)

Let $U \geq 0$ be a traveling wave and assume that U is continuous. Then $U \in C^1(\mathbb{R})$, U is strictly positive and

$$-\chi(\rho_x \star U)(x) < c \qquad \text{for all } x \in \mathbb{R}.$$

Existence of non-sharp smooth traveling waves waves

This last result is a recent work with Pierre Magal and Min Zhao[5].

Theorem (Existence of a continuous traveling wave)

We assume that

$$c \geq 2\sqrt{\sigma^2 \hat{\chi} (1 + \hat{\chi})}.$$

There exists a traveling wave $u(t, x) = U(x - ct)$ with a continuous profile $x \to U(x)$ is continuously differentiable and strictly decreasing, and

$$\lim_{x \to -\infty} U(x) = 1, \text{ and } \lim_{x \to +\infty} U(x) = 0, \tag{3}$$

and satisfies traveling wave problem

$$-c\, U' - \chi (UP')' = U(1 - U), \text{ on } \mathbb{R}, \tag{4}$$

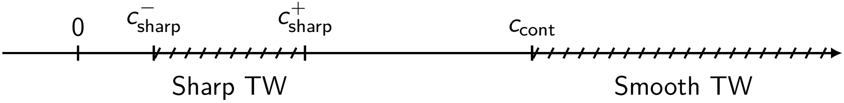
where

$$P - \sigma^2 P'' = U, \text{ on } \mathbb{R}. \tag{5}$$

[5] Quentin Griette, Pierre Magal, and Min Zhao. *Traveling waves with continuous profile for hyperbolic Keller-Segel equation*. 2022.

Recall $\hat{\chi} = \dfrac{\chi}{\sigma^2}$.

- Suppose $0 < \hat{\chi} \leq \bar{\chi}$ (with $\bar{\chi} \approx 1.045$). For $c \in \left(\dfrac{\sigma \hat{\chi}}{2 + \hat{\chi}}, \dfrac{\sigma \hat{\chi}}{2} \right) =: (c^-_{\text{sharp}}, c^+_{\text{sharp}})$: existence of discontinuous waves
- For any $c \geq c_{\text{cont}} := 2\sqrt{\sigma^2 \hat{\chi} \left(1 + \hat{\chi}\right)} > c^+_{\text{sharp}}$ there exists a continuous traveling wave (no restriction on $\hat{\chi}$).



So there is a gap between the two “zones of existence”.

Uniqueness of traveling waves and non-existence of waves: open questions.

PART II:

Speed-up of traveling waves by negative chemotaxis

joint work with Chris Henderson and Olga Turanova[6].

[6] Quentin Griette, Christopher Henderson, and Olga Turanova. "Speed-up of traveling waves by negative chemotaxis". *J. Funct. Anal.* 285.10 (2023), Paper No. 110115, 67. DOI: 10.1016/j.jfa.2023.110115.

The repulsive chemotaxis model

We aim at studying the model:

$$\begin{cases} U_t + \chi(V_x U)_x = U_{xx} + U(1-U), \\ -dV_{xx} = U - V, \end{cases}$$

and more specifically the traveling waves

$$\begin{cases} -\bar{c}U_x + \chi(V_x U)_x = U_{xx} + U(1-U), \\ -dV_{xx} = U - V, \end{cases}$$

with the conditions :

$$U(-\infty) = 1, \qquad U(+\infty) = 0, \qquad V \in L^\infty(\mathbb{R}).$$

χ **will be negative:** $-\chi > 0$**.**

$$\begin{cases} -\bar{c}U_x + \chi(V_xU)_x = U_{xx} + U(1-U), \\ -dV_{xx} = U - V, \end{cases}$$

The speed in the absence of advection is $c_{KPP} = 2$.

In is known[7] that when $-\chi > 0$, the minimal speed $\bar{c}_{\chi,d}$ satisfies $\bar{c}_{\chi,d} \geq 2$ (no slow down by repulsive chemotaxis). More precisely, it has been proved that :

1. when $d, \frac{-\chi}{d} \ll 1$ then $\bar{c}_{\chi,d} = 2$
2. when $1 \ll -\chi \ll d$ then $\bar{c}_{\chi,d} \approx \frac{|\chi|}{2\sqrt{d}}$ (so there is a speed-up)

We prove that[8]

4. when $d \ll -\chi$ then $\bar{c}_{\chi,d} \gtrapprox c_{pm,\varepsilon}\sqrt{-\chi}$ ($-\chi$ may remain finite)
5. when $d \ll -\chi$ then $\bar{c}_{\chi,d} \gtrapprox c_{pm,\varepsilon}\sqrt{-\chi}$

[7] Christopher Henderson. "Slow and fast minimal speed traveling waves of the FKPP equation with chemotaxis". *J. Math. Pures Appl. (9)* 167 (2022), pp. 175–203. DOI: 10.1016/j.matpur.2022.09.004.

[8] Griette, Henderson, and Turanova, "Speed-up of traveling waves by negative chemotaxis".

The rescaling

$$\begin{cases} -\bar{c}U_x + \chi(V_x U)_x = U_{xx} + U(1-U), \\ -dV_{xx} = U - V, \end{cases}$$

We introduce a rescaling of the unknown functions

$$u(x) = U(x\sqrt{-\chi}), \qquad v(x) = V(x\sqrt{-\chi}),$$

so the new functions solve the equation

$$\begin{cases} -\dfrac{\bar{c}}{\sqrt{-\chi}} u_x - (v_x u)_x = \dfrac{1}{-\chi} u_{xx} + u(1-u), \\ -\dfrac{d}{-\chi} v_{xx} = u - v, \end{cases}$$

We look at $-\chi$ and d as varying parameters which may go to $+\infty$ or 0.

Limit equations: Porous medium

$$\begin{cases} -cu_x - (v_x u)_x = \dfrac{1}{-\chi} u_{xx} + u(1-u), \\ -\nu v_{xx} = u - v, \end{cases}$$

with $c = \frac{\bar{c}}{\sqrt{-\chi}}$ and $\nu = \frac{d}{-\chi}$.

$\rightsquigarrow$ **First case: porous medium medium limit.** Up to extraction,

$$\nu \to 0, \qquad -\chi \to \frac{1}{\varepsilon}, \text{ with } \varepsilon \in [0, +\infty).$$

The limit equation is local (porous medium-type) and the minimal speed is known:

$$c^*_{\mathrm{pm},\varepsilon} = \begin{cases} \frac{1}{\sqrt{2}} + \sqrt{2\varepsilon}, & \text{if } \varepsilon < \frac{1}{2}, \\ 2\sqrt{\varepsilon}, & \text{if } \varepsilon \geq \frac{1}{2}. \end{cases}$$

$$\begin{cases} -cu_x - (v_x u)_x = \dfrac{1}{-\chi} u_{xx} + u(1-u), \\ -\nu v_{xx} = u - v, \end{cases}$$

with $c = \frac{\bar{c}}{\sqrt{-\chi}}$ and $\nu = \frac{d}{-\chi}$.

$\rightsquigarrow$ **Second case: hyperbolic limit.** Up to extraction,

$$\nu \to \text{positive constant}, \ \nu, \qquad -\chi \to 0.$$

The limit equation is non-local (hyperbolic type), we some properties but the minimal speed is unknown.
We derive a **universal positive lower bound** for the speed of the hyperbolic problem, among other things.

Results. I: the porous medium limit

$c^*_{\chi,\nu}$ is the minimal speed for the rescaled equation; $c^*_{\mathrm{pm},\varepsilon}$ is the minimal speed of the porous medium limit.

Theorem

Fix any $\varepsilon \geq 0$.

(i) *The minimal speeds have the asymptotics:*

$$\liminf_{-\chi \to \frac{1}{\varepsilon}, \nu \to 0} c^*_{\chi,\nu} \geq c^*_{pm,\varepsilon},$$

(ii) *Consider any sequence* $(\chi_n \to \chi, \nu_n \to \nu)$ *and traveling wave solutions* (c_n, u_n, v_n) *to the rescaled equation. If* $\limsup c_n < +\infty$, *then up to shifting* (u_n, v_n) *so that*

$$\min_{x \leq 0} u_n(x) = u_n(0) = \delta \in (0,1),$$

there exists a (c, u) *solution to the porour medium KPP equation and a subsequence* n_k *with*

$$c_{n_k} \to c, \qquad u_{n_k} \to u \text{ in } L^\infty_{loc}, \qquad \text{and } v_{n_k} \rightharpoonup v \text{ in } H^1_{loc}.$$

Results. IIa: hyperbolic waves, weak notion

We developed a **new weak notion of hyperbolic traveling waves** to deal with the limit of the rescaled equation in a natural way.

$$\begin{cases} -(c+v_x)u_x = u\left(1+\dfrac{v-u}{\nu}-u\right), \\ -\nu v_{xx} = u - v, \end{cases} \tag{6}$$

Given $c > 0$, $u \in L^\infty(\mathbb{R})$ and $v \in W^{2,\infty}(\mathbb{R})$. We denote

$$\mathcal{Z} = \{x \,:\, c + v_x(x) = 0\}.$$

We say that (c, u, v) is a traveling wave if v solves the second equation in (6) on $\mathbb{R}$, $u \in C^1_{loc}(\mathcal{Z}^c)$ and solves (6) on $\mathcal{Z}^c$, and we have

$$u\left(1+\frac{v-u}{\nu}-u\right) = 0$$

on $Int(\mathcal{Z})$.

The equation allows us to gain some regularity on the weak solutions.

Proposition

Let (c, u, v) be a weak hyperbolic traveling wave. Suppose that u is nonconstant, that is, both $\{u > 0\}$ and $\{u < 1\}$ have positive measures. Then $c > 0$ and there are only two possibilities:

(i) $\mathcal{Z} = \varnothing$. *In that case, (u, v) is a classical solution of the system.*

(ii) $\mathcal{Z} = \{x_0\}$ *consists of a single point. In that case, u has a single jump discontinuity at x_0, with $\{u > 0\} = (-\infty, x_0)$. Moreover $u \in C^\infty_{loc}(\mathbb{R}\backslash\{x_0\})$ and u satisfies, at the jump,*

$$u(x_0^-) = \frac{\nu + v(x_0)}{\nu + 1}.$$

Results. IIc: hyperbolic waves, estimates on the speed.

It is known that, for any traveling wave (either sharp discontinuous or smooth)

$$\sup_{x\in\mathbb{R}} -v_x(x) \le c.$$

Proposition

Fix $-\chi \in (-\chi_0, +\infty]$, $0 < \nu_m < \nu_M$ and $C_M > 0$. Let (c, u, v) be either a solution of the rescaled or hyperbolic equation, with $c \in [0, C_M]$. If

$$u(0) \le \frac{\nu}{\nu+1},$$

then there exists $\theta > 0$ and $C > 0$ depending only on $-\chi_0$, ν_m, ν_M and C_M with

$$u(x) \le Cu(0)e^{-\theta x} \text{ for all } x \ge 0.$$

Corollary

Fix $0 < \nu_m < \nu_M$. There exists $\underline{c}$, depending only on ν_m and ν_M, such that for any (c, u, v) solution to the hyperbolic model with $\nu \in [\nu_m, \nu_M]$, we have

$$c \ge \underline{c}.$$

Results. II: hyperbolic waves, limit speed.

Theorem

Let $\nu_{hyp} > 0$.

(i) *The minimal speeds have the asymptotics:*

$$\liminf_{-\chi \to +\infty, \nu \to \nu_{hyp}} c = \underline{c}(\nu_{hyp}) > 0.$$

(ii) *Consider any sequence* $(-\chi_n \to +\infty, \nu_n \to \nu_{hyp})$ *and traveling wave solutions* (c_n, u_n, v_n) *to the rescaled equation. If* $\limsup c_n < +\infty$, *then up to shifting* (u_n, v_n) *so that*

$$\min_{x \leq 0} u_n(x) = u_n(0) = \delta, \text{ with } 0 < \delta < \frac{\nu_{hyp}}{\nu_{hyp} + 1},$$

there exists a solution (c, u, v) *to the hyperbolic model and a subsequence* n_k *with*

$$c_{n_k} \to c, \qquad u_{n_k} \to u \text{ in } C^1(\mathcal{Z}^c), \qquad \text{and } v_{n_k} \to v \text{ in } C^2(\mathcal{Z}^c) \cap W^{2,\infty}_{w-*}(\mathbb{R}),$$

where either $\mathcal{Z} = \varnothing$ *or* $\mathcal{Z} = \{x_0\}$.

Results IIIa: upper bound for the hyperbolic limit

We obtain upper bounds for the minimal speed by constructing the solutions of the rescaled equation that converge to a specific solution of the hyperbolic and porous medium models.

Theorem

Fix $\nu_{hyp} > 0$ and any sequence $-\chi_n \to +\infty$ and $\nu_n \to \nu_{hyp}$. There exists a solution (c_n, u_n, v_n) of the rescaled equation and a solution (c, u, v) to the hyperbolic model such that

(i) *Both (u_n, v_n) and (u, v) are decreasing in x,*

(ii) *u is sharp and has a jump discontinuity at $x = 0$,*

(iii) *Up to the extraction of a subsequence, we have*

$$\lim(c_n, u_n, v_n) = (c, u, v)$$

in the topology of $\mathbb{R}$, $C^1(\mathbb{R}\backslash\{0\})$, and $W^{2,\infty}(\mathbb{R})$ weak-$*$

Results IIIb: upper bound for the porous medium limit

We show that the hyperbolic discontinuous wave converge as $\nu \to 0$ to a sharp (and consequently minimal speed) wave for the porous medium model and use a double limit to show that the asymptotic minimal speed in the porous medium case is the expected one.

Theorem

Consider the decreasing family of sharp discontinuous traveling waves to the hyperbolic model with parmeter ν constructed previously, (c, u, v). Then

$$\lim_{\nu \to 0} (c, u) = \left(\frac{1}{\sqrt{2}}, u_{pm} \right),$$

where u_{pm} is the unique minimal speed traveling wave to the porous medium-KPP equation with $\{u_{pm} > 0\} = (-\infty, 0)$.

Then thanks to a careful double limit, we obtain the corollary

Corollary

We have

$$\lim_{-\chi \to \infty, \nu \to 0} c^*_{\chi,\nu} = c^*_{pm,0} = \frac{1}{\sqrt{2}}.$$

Thank you !

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The Cauchy problem: Proof of the well-posedness

The proof of the well-posedness is technical. The main steps of the proof are as follows:

FIRST STEP: We fix $\mathcal{U} \subset \mathbb{R}$ a conull set, $u_0 \in \mathcal{L}^\infty(\mathcal{U})$ and focus on the system obtained by the change of variables $w(t,x) := u(t, \Pi(t,0;x))$ and $\hat{p}(t,x) := p(t, \Pi(t,0;x))$. Here $\Pi(t,s;x)$ is the flow of the characteristic equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\Pi(t,s;x) = -\chi p_x\big(t, \Pi(t,s;x)\big), \\ \Pi(s,s;x) = x. \end{cases}$$

Then we can show that $(w, \hat{p})$ is a fixed-point of the operator

$$\mathcal{T}_{\mathcal{U}}^{\tau}[u_0](w,\hat{p})(t,x) = \begin{pmatrix} u_0(x)\exp\left(\int_0^t 1 + \hat{\chi}\hat{p}(l, \Pi(l,0;x)) - (1+\hat{\chi})w(l,x)\mathrm{d}l\right) \\ \int \frac{1}{2\sigma} e^{-|x-\Pi(t,0;z)|/\sigma} u_0(z) e^{\int_0^t 1 - w(l,z)\mathrm{d}l} \mathrm{d}z \end{pmatrix}^T$$

For $\tau^{\mathcal{U}}(u_0)$ sufficiently small this operator acting on the adequate set of functions over $[0, \tau^{\mathcal{U}}(u_0)]$ is a contraction. $\tau^{\mathcal{U}}(u_0)$ depends only on $\|u_0\|_{\mathcal{L}^\infty(\mathcal{U})}$.
However, we don't have a semigroup property with this formulation.

The Cauchy problem: Proof of the well-posedness

SECOND STEP: In order to get a semi-group property, we need to go back to the original formuation. **However, the characteristics may not have preserved the conull set $\mathcal{U}$ on which u_0 is defined.** Therefore we "go back" to $u_0 \in L^\infty(\mathbb{R})$ and apply the construction of Step 1 to a particular choice of $\mathcal{U} \subset \mathbb{R}$ and $\hat{u}_0 \in \mathcal{L}^\infty(\mathcal{U})$ such that $\|\hat{u}_0\|_{\mathcal{L}^\infty(\mathcal{U})} = \|u_0\|_{L^\infty(\mathbb{R})}$. It can be shown that the L^∞-class of $u(t,x)$ is independent of the choice of $\mathcal{U}$ and $\hat{u}_0$.
Now $u(t,x)$ satisfies a semigroup property.

THIRD STEP: By the semigroup property and classical arguments, there exists a maximal time of existence $\tau^*(u_0) \in (0,+\infty]$, such that

$$\text{either } \tau^*(u_0) = +\infty \text{ or } \liminf_{t\to\tau^*(u_0)^-} \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} = +\infty.$$

We actually have a stronger result: that there exists a conull set $\mathcal{U}$ and a real function $u_0 \in \mathcal{L}^\infty(\mathcal{U})$ such that the $w(t,\cdot) := u(t,\Pi(t,0;\cdot))$ is the L^∞ class of $\hat{w}(t,\cdot)$, where $(\hat{w},\hat{p})$ is the unique fixed point of the original map $\mathcal{T}_{\mathcal{U}}^{\tau}[u_0]$ **for all** $0 < \tau \le \tau^*(u_0)$.
The L^1_η continuity of $t \mapsto u(t,\cdot)$ can be obtained from this property.

FOURTH STEP: The remaining two properties, namely, the continuity of the map $u_0 \mapsto u(t,\cdot)$ for the $L^\infty(\mathbb{R}) - L^1_\eta(\mathbb{R})$ topologies and the equivalence between the solutions to the fixed-point problem and the integrated solutions are done independently by ad-hoc methods. We refer to the paper for details. This finished the proof of the Theorem.

Now we have in our hands a solution $u(t,\cdot) \in L^\infty(\mathbb{R})$ which is uniquely defined on $\big[0, \tau^*(u_0)\big) \times \mathbb{R}$ and continuous for the $L^1_\eta(\mathbb{R})$-topology. Moreover, the field $p(t,x)$ is well-defined and continuous for the $W^{1,\infty}(\mathbb{R})$ topology and has bounded second derivative:

$$\|p_{xx}(t,\cdot)\|_{L^\infty(\mathbb{R})} < +\infty.$$

The flow of the characteristic curves $\Pi(t,s;x)$ is well-defined and Lipschitz continuous with respect to x.

Sketch of the proof of existence of a sharp traveling wave

The traveling wave equation is:

$$(-c-\chi P'(z))U'(z) = U(z)\big(1+\hat{\chi}P(z)-(1+\hat{\chi})U(z)\big),$$

where $P(z)=(U\star\rho)(z)$. We look for a profile

$$U(z)=0 \text{ if } z>0, U(z)>0 \text{ if } z<0.$$

To have a discontinuity on the profile the equation must be degenerate therefore

$$c=-\chi P'(0).$$

The equation under consideration is

$$\chi(P'(0)-P'(z))U'(z) = U(z)\big(1+\hat{\chi}P(z)-(1+\hat{\chi})U(z)\big).$$

Sketch of the proof of existence of a sharp traveling wave

$$\chi(P'(0) - P'(z))U'(z) = U(z)\big(1 + \hat{\chi}P(z) - (1+\hat{\chi})U(z)\big).$$

We introduce a change of variable:

$$\begin{cases} \tau'(t) = \chi\big(P'(0) - P'(\tau(t))\big) \\ \tau(0) = -1. \end{cases}$$

Then $\mathcal{U}(t) = U(\tau(t))$ satisfies:

$$\mathcal{U}'(t) = \mathcal{U}(t)\big(1 + \hat{\chi}P(\tau(t)) - (1+\hat{\chi})\mathcal{U}(t)\big),$$

therefore

$$\mathcal{U}(t) = \frac{1}{(1+\hat{\chi})\int_{-\infty}^{t} \exp\left(-\int_{l}^{t} 1 + \hat{\chi}P(\tau(s))\mathrm{d}s\right)\mathrm{d}l}$$

We look for U as a fixed-point of

$$\mathcal{T}(U)(z) := \mathcal{U}(\tau^{-1}(z))$$

Sketch of the proof of existence of a sharp traveling wave

$$\chi(P'(0) - P'(z))U'(z) = U(z)\big(1 + \hat{\chi}P(z) - (1 + \hat{\chi})U(z)\big).$$

Admissible profiles U are:

1. continuous,
2. valued in $[0, 1]$, and $\lim_{z \to 0^-} U(z) \geq \frac{2}{2+\hat{\chi}}$,
3. sharp, *i.e.* $U(z) = 0$ for all $z \geq 0$,
4. non-increasing.

It can be shown that this set $\mathcal{A}$ of admissible profiles is invariant by $\mathcal{T}$. Moreover $\mathcal{T}$ is continuous and compact on $\mathcal{A}$ for the topology induced by

$$\|U\|_\eta := \sup_{z<0} e^{\eta z}\sqrt{-z}\,U(z).$$

Since $\mathcal{A}$ is convex, the Schauder fixed-point theorem concludes that $\mathcal{T}$ has a fixed-point on $\mathcal{A}$.

"A Billiard Problem in Nonlinear Dissipative Systems"

Shin-Ichiro Ei (Hokkaido University, Japan)

The motion of camphor discs in a square domain is considered. Different motions from a usual Billiard problem are observed such as the existence of a stable limit cycle. This talk is mainly done according to the content of the monograph by Miyaji, E. and Mimura. The interaction of elliptic camphor discs is also mentioned.

A Billiard Problem in Nonlinear Dissipative Systems

-for the memory of Prof. Masayasu Mimura-

Shin-Ichiro Ei
Hokkaido University
Sapporo, Japan

Short summary of careers of Mimura sensei and I

Mimura sensei

-1973, Kyoto Univ.
1970-1980, Konan Univ.

1980-1993, Hiroshima Univ. ⊃

1993-1998, Tokyo Univ. ⊂

1998-2004, Hiroshima Univ.
2004-2017, Meiji Univ.
2017- , Musashino Univ., Meiji Univ.

Tsujikawa, Nakaki, Kan-on, I,
Tohma, Kuwamura, …
Nagayama, Izuhara, …
:members related to ReaDiNet

Sorry to other members !

I

Staffs:
Tomoeda, Ito, Matano, Kobayashi,
Nishiura, Ogawa, …

1982-1992, Hiroshima Univ.

1992-2004, Yokohama City Univ.

2004-2014, Kyushu Univ.
2014 - , Hokkaido Univ.

Sensei looked very busy,
but enjoyed research,
softball and tennis.
Many students in Mimura Lab.

Mimura’s Lab. in Hiroshima


FUJICOLOR HR 84


'83 6 15


'88 6 11

In seminars

Simply show simple things.
(Simple things should be shown in a simple way)

sometimes

Explain intuitively (and understandably). frequently

What is interesting ? What is the motivation ? absolutely
(What is the salse point ?)

overwhelmed by his active power and outstanding insight

Spiritually mooring

The first paper for me

Spatial Distribution of Rapidly Dispersing
Animals in Heterogeneous Environments

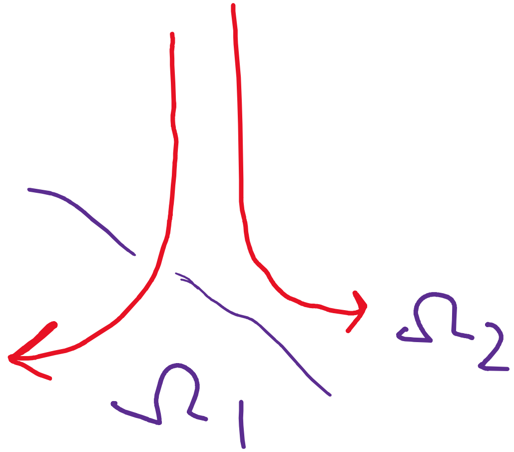
Article · January 1984
DOI: 10.1007/978-3-642-87422-2_33

by Nanako Shigesada

1984

1. Introduction

Ecological models incorporating spatial heterogeneity of habitats are of profound importance in understanding the movements of organisms and their effects on the stability of spatial distributions of populations under natural circumstances. Equations describing the time development of the spatial distribution of a population in a heterogeneous environment fundamentally involves two terms, dispersal and growth, which are both functions of space. There have been several distinct approaches to the analysis of such models



The first paper which Mimura sensei gave me !
Related to transient and asymptotic motions of solutions for a RD with heterogeneous media.

⇨ The motivation fixed my research direction

PhD in 1987

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11. S.-I. Ei, M. Mimura and M. Nagayama, Interacting Spots in reaction diffusion systems, DCDS 14 (2006), 31-62.

12. Xinfu Chen, S.-I. Ei and M. Mimura, SELF--MOTION OF CAMPHOR DISCS -MODEL AND ANALYSIS-, NETWORKS AND HETEROGENEOUS MEDIA Volume 4, Number 1 (2009), 1-18.

13. S.-I. Ei, H. Izuhara and M. Mimura, Infinite dimensional relaxation oscillation in aggregation-growth systems, Discrete and Continuous Dynamical Systems, Series B, 17 (2012) 1859 - 1887.

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15. Shin-Ichiro Ei, Masayasu Mimura and Tomoyuki Miyaji, Reflection of a self-propelling rigid disk from a boundary, DCDS-A 2021, 14(3): 803-817
Special issue on recent topics in material, computer and life sciences. doi: 10.3934/dcdss.2020229

Short history of an ongoing joint work

Tomoyuki Miyaji, Shin-Ichiro Ei
Masayasu Mimura

A Billiard Problem in Nonlinear Dissipative Systems

– Monograph –

August 9, 2021

springer

preface

Classical billiard problem has been extensively studied in the community of mathematics. However, this book is concerned with billiard motion of a self-propelling disk in a nonlinear dissipative system on a rectangular pool. As an example of self-propelling disk, a camphor disk floating on water vessel is well-known in nonlinear science. The laboratory experiment demonstrates that, unlike classical billiards, it has the following properties:

(i) It reflects without collision at the boundary, and
(ii) the angle of reflection is greater than that of incidence, which is totally different from elastic reflection.

Motion of pulses

Joint works with Mimura sensei for pulse/front dynamics

. M. Mimura, K. Sakamoto and S.-I. Ei, Singular perturbation problems to a combustion equation in very long cylindrical domains, AMS/IP Studies in Advanced Math. vol. 3(1997), 75-84.

. S.-I. Ei, R. Ikota and M. Mimura, Segregating pattern problem in competition-diffusion systems, J. Interfaces and Free Boundaries vol. 1(1999), 57-80.

. S.-I. Ei, M. Mimura and M. Nagayama,
Pulse-pulse interaction in reaction-diffusion systems,
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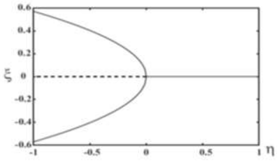
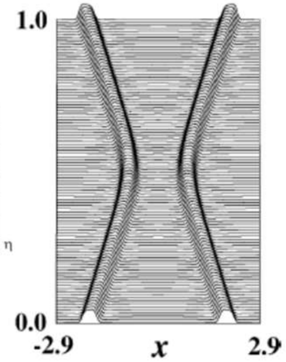
. S.-I. Ei, M. Mimura and M. Nagayama,
Interacting Spots in reaction diffusion systems, DCDS 14 (2006), 31-62.

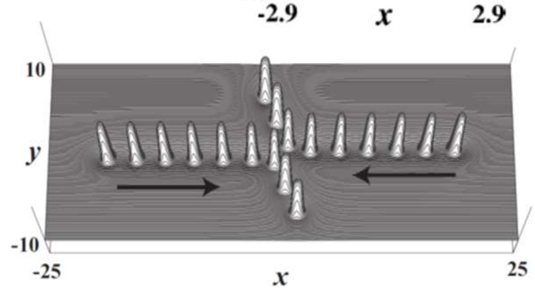
. Xinfu Chen, S.-I. Ei and M. Mimura,
SELF--MOTION OF CAMPHOR DISCS -MODEL AND ANALYSIS-,
NETWORKS AND HETEROGENEOUS MEDIA Volume 4, Number 1 (2009), 1-18.

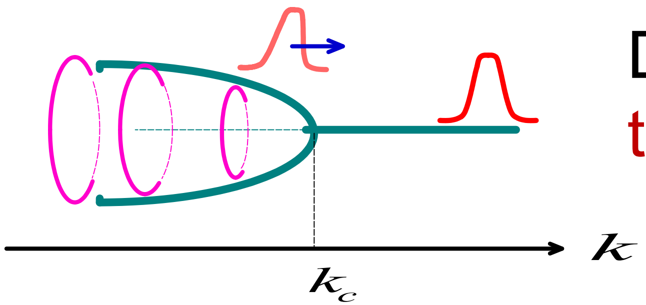
1D reflec.





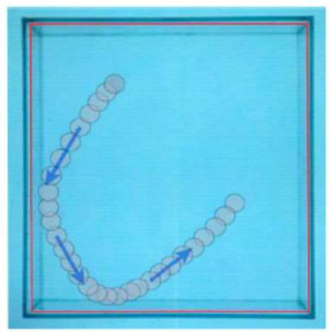
2D reflec.





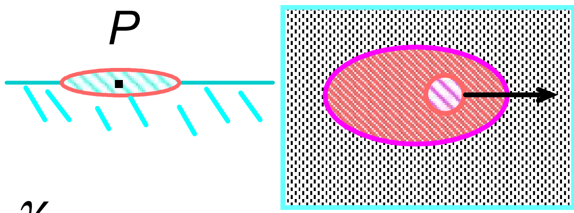
Drift bifurcation of
traveling pulse, spot

Pitch-fork type bifurcation
diagram of pulses, spots



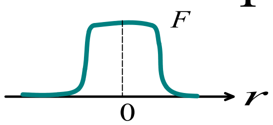
Camphor Layer Model(樟脳)

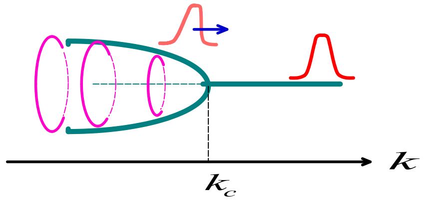
- Nagayama et.al.00



$$\begin{cases} u_t = D\Delta u - au + F(|x-P|), \\ \ddot{P} = \nabla_x \gamma(u)|_{x=P} - \mu \dot{P}, \end{cases}$$

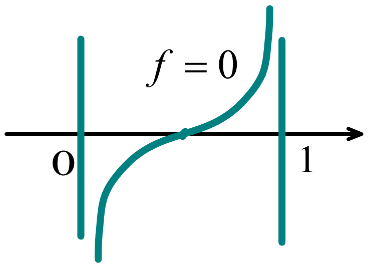
$$\gamma(u) = \frac{\gamma_0}{1+cu}$$

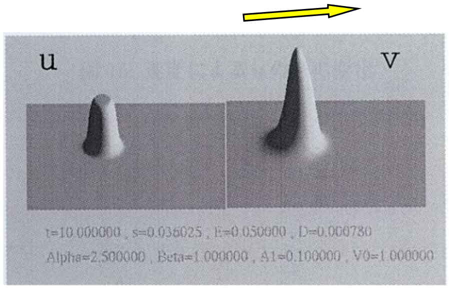




- Mimura et.al.01

$$\begin{cases} \tau\varepsilon u_t = \varepsilon^2 \Delta u + f(u,v) - < f >, \\ v_t = D\Delta v + au - \gamma v, \end{cases}$$



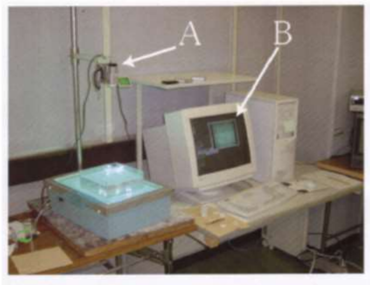


(Koya 00)

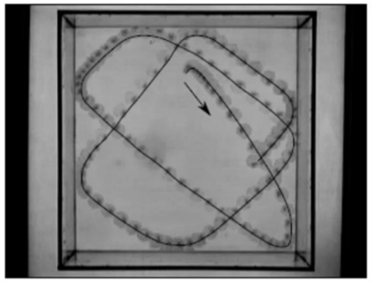
$$< f > = \frac{1}{|\Omega|}\int f\,dx, \quad f(u,v) = u(1-u)(u-a(v)), \quad a(v) = \frac{1+\tanh v}{2}$$

Experiment for camphor disk

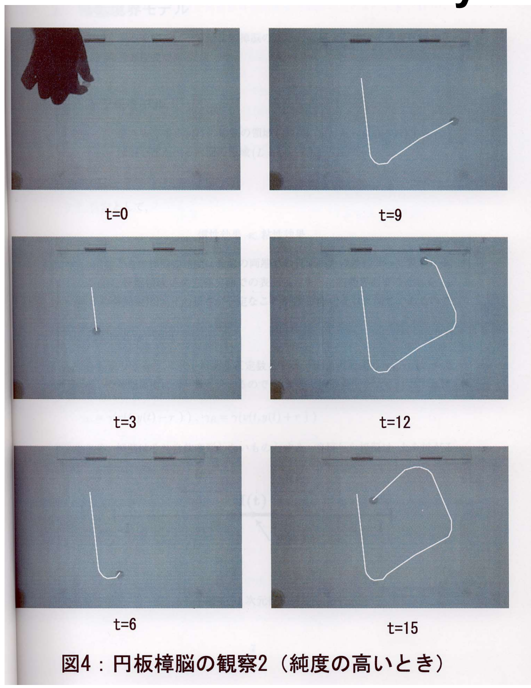
By Kanda (02, Hiroshima Univ.)

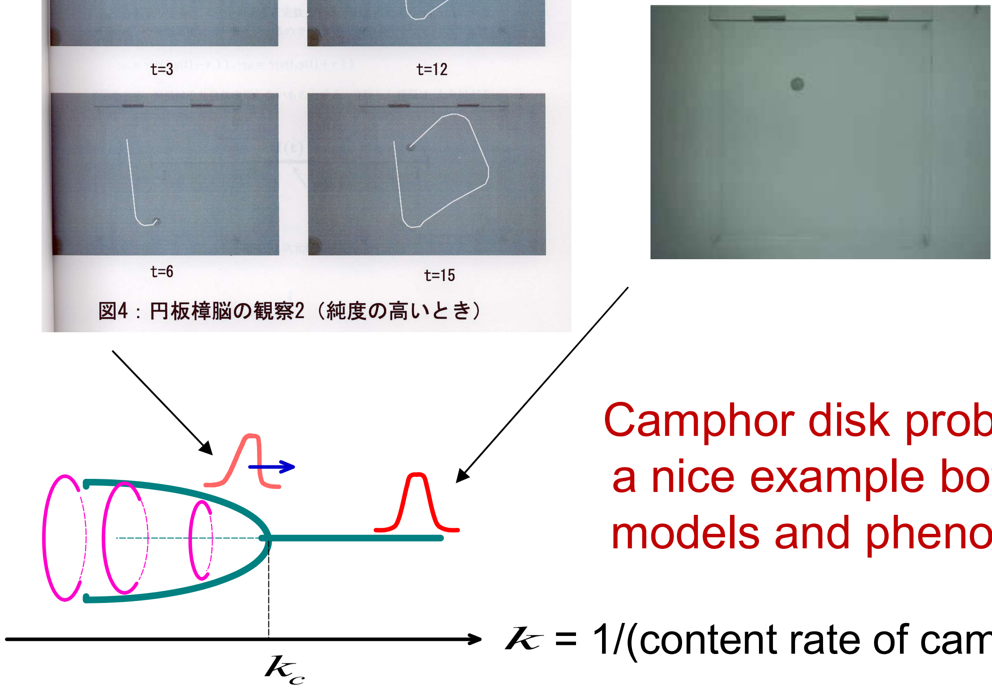


Recorded by the video camera at A and monitored by the display at B to produce a movie.



A trajectory of a camphor disk of an experiment

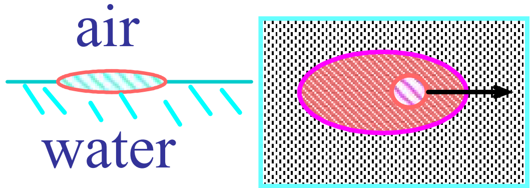




k = 1/(content rate of camphor)

Camphor disk problem is a nice example both from models and phenomena !

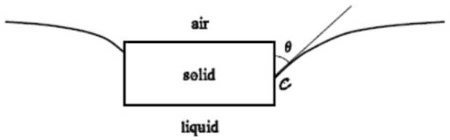
Moving boundary (MB) model for camphor disk

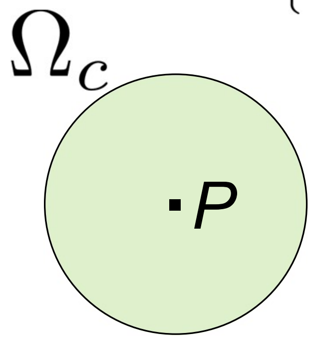


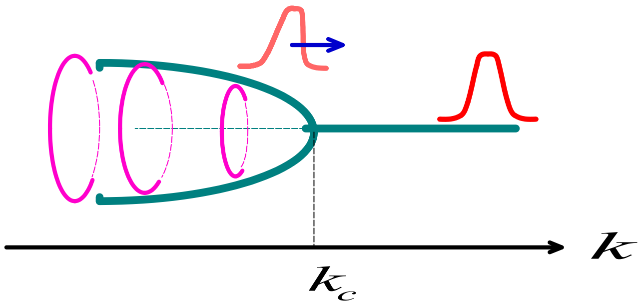
$$\begin{cases} \dfrac{dP}{dt} &= \displaystyle\int_{\partial\Omega_c} \gamma(u)\boldsymbol{n}dl, \\ u_t &= \Delta u - u + f(\boldsymbol{x}, P), \end{cases}$$

Chen, E, Mimura 2009

$$f(\boldsymbol{x}, P) = \begin{cases} f_0, & \boldsymbol{x} \in \overline{\Omega}_c, \\ 0, & \boldsymbol{x} \in \Omega \backslash \overline{\Omega}_c, \end{cases}$$







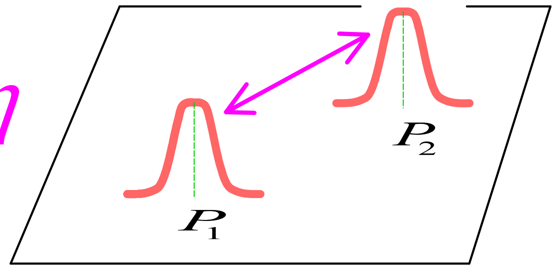
WS in Lorentz center 2003?
Drift bifurcation was shown by Chen

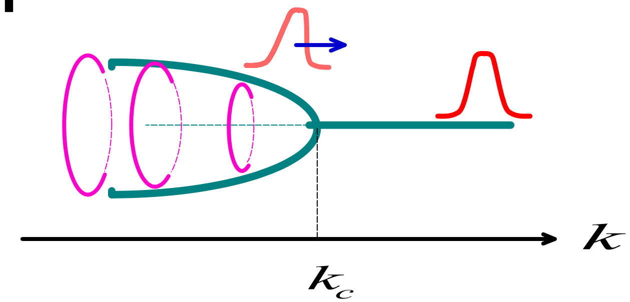
First rigorous result
for 2D reflec.

MB model which can be rigorously treated.

Interaction of camphor disks

$$k = k_c + \eta$$



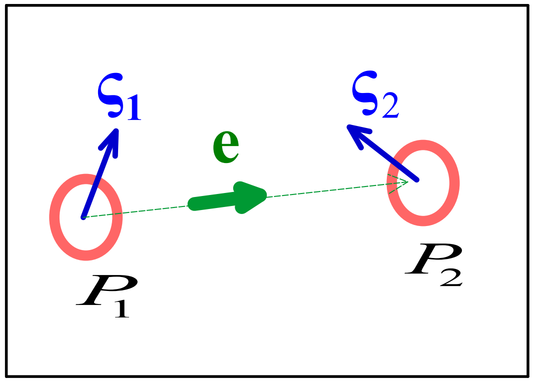


MB model

$$M_0, M'_0 > 0 \Rightarrow$$ Repulsive interaction

$$\begin{cases} \dot{P}_1 = \varsigma_1 - \dfrac{M_0}{\sqrt{h}} e^{-\alpha h} \mathbf{e}, \\ \dot{P}_2 = \varsigma_2 + \dfrac{M_0}{\sqrt{h}} e^{-\alpha h} \mathbf{e}, \\ \dot{\varsigma}_1 = -\nabla W(\varsigma_1) - \dfrac{M'_0}{\sqrt{h}} e^{-\alpha h} \mathbf{e}, \\ \dot{\varsigma}_2 = -\nabla W(\varsigma_2) + \dfrac{M'_0}{\sqrt{h}} e^{-\alpha h} \mathbf{e}, \end{cases}$$

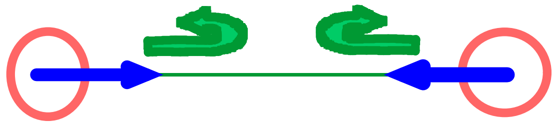
$$h = |P_2 - P_1|, \mathbf{e} = \frac{|P_2 - P_1|}{h}$$



Special Cases

- On a line

$$P_j = (p_j, 0), \boldsymbol{\varsigma}_\mathbf{j} = (\varsigma_j, 0)$$

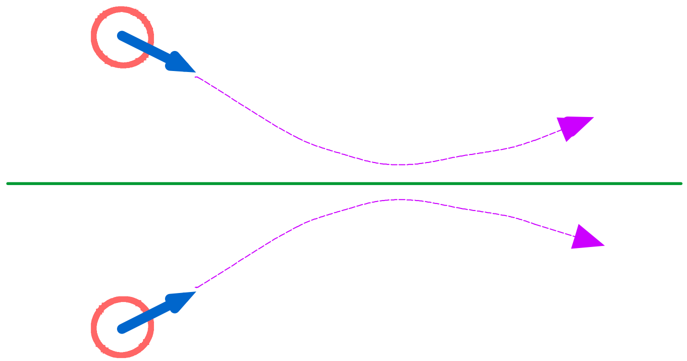


(E. Mimura, Nagayama 02)

- Neuman boundary

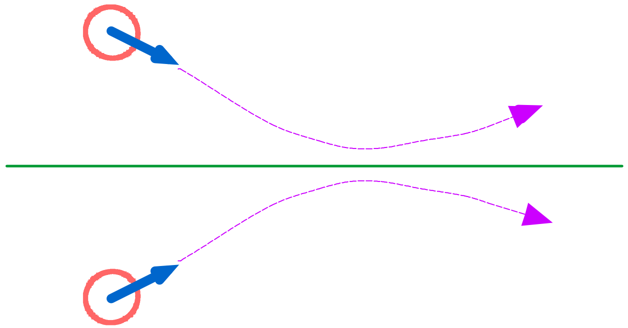
$$P_1 = (p, q), \boldsymbol{\varsigma}_1 = (\varsigma, \xi), P_2 = (p, -q), \boldsymbol{\varsigma}_2 = (\varsigma, -\xi),$$

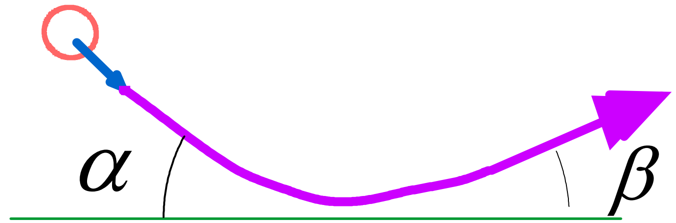
(Matsumoto 02)



Dynamics of ODE

$$P_1 = (p, q), \boldsymbol{\varsigma}_1 = (\varsigma, \xi), P_2 = (p, -q), \boldsymbol{\varsigma}_2 = (\varsigma, -\xi),$$





$$\begin{cases} \dot{p} = \varsigma, \\ \dot{q} = \xi + \dfrac{M_0}{\sqrt{2q}} e^{-2\alpha q}, \\ \dot{\varsigma} = -(M_1 |\boldsymbol{\varsigma}|^2 - M_2 \eta)\varsigma, \\ \dot{\xi} = -(M_1 |\boldsymbol{\varsigma}|^2 - M_2 \eta)\xi + \dfrac{M'_0}{\sqrt{2q}} e^{-2\alpha q}, \end{cases}$$

$$\beta = T(\alpha)$$

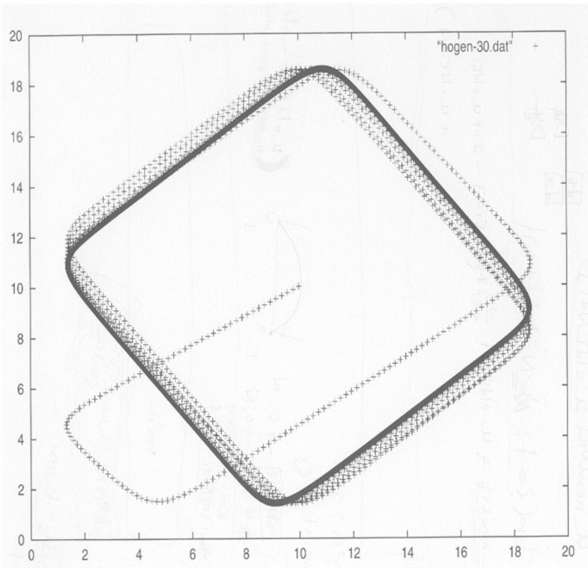
$\alpha > \beta$ (?)

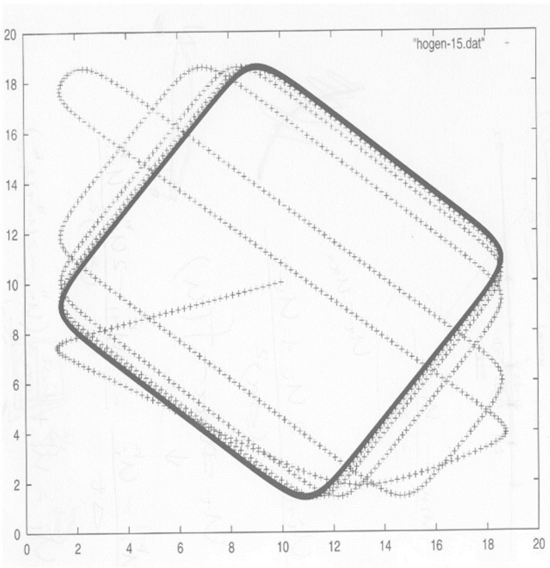
$(\alpha \geq \beta)$ **Mimura et.al**

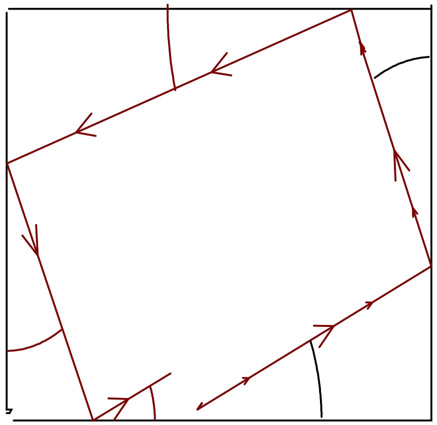
Meaning?

difference from usual billiard problem ?

Dynamics in a square region







Mimura sensei discovered the stable limit cycle

c.f. Billiard problem (Generically the region is densely filled by orbits)

Experiment in Hiroshima
by Mimura and Lab. students 2002

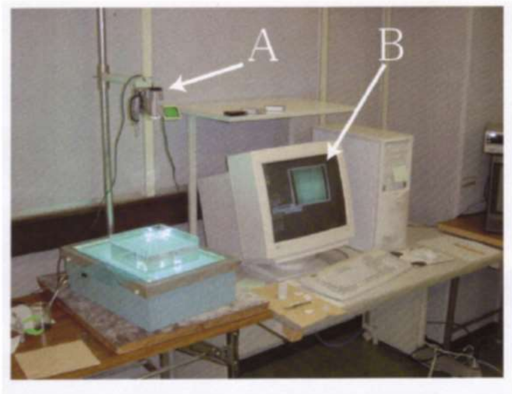


Fig. 1.1 The experimental facility by Kanda[4]. The experiment was recorded by the video camera at A and monitored by the display at B to produce a movie.

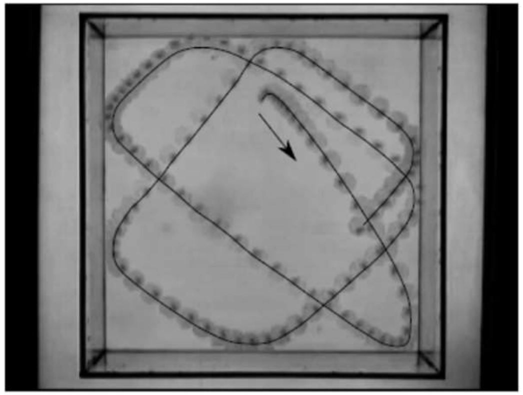


Fig. 1.2 A trajectory of a camphor disk of an experiment by [4].

Complicated motions of camphor disk

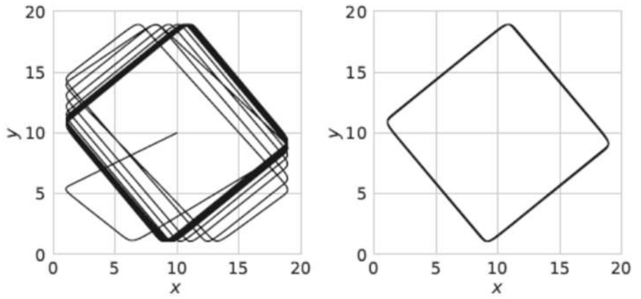


Fig. 1.9 Trajectories of the center of the disk of moving boundary model on square . Left: $0 \le t \le 10^4$, Right: $9.5 \times 10^3 \le t \le 10^4$.

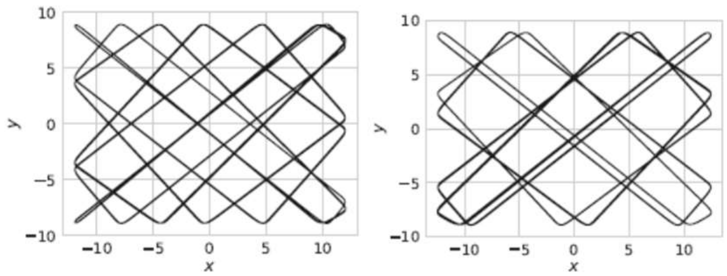
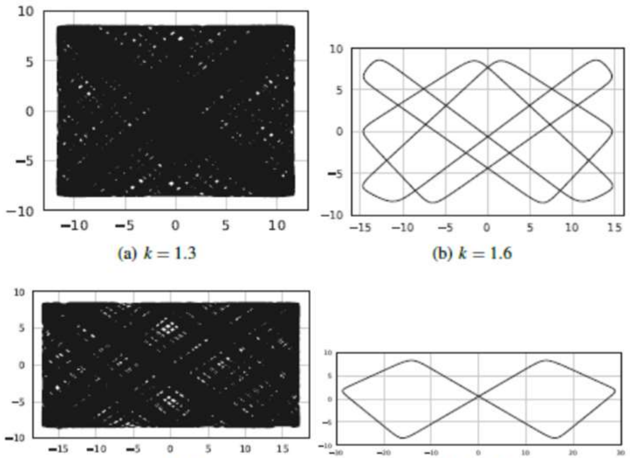


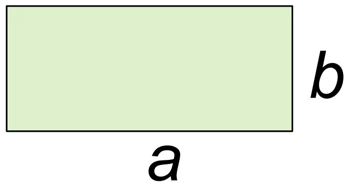
Fig. 1.11 Complicated periodic orbits for (1.11) on rectangular domains with $k = 1.3$(left) and $k = 1.35$ (right).



(a) $k = 1.3$ (b) $k = 1.6$

(c) $k = 1.85$ (d) $k = 3.0$

$$k = a/b$$



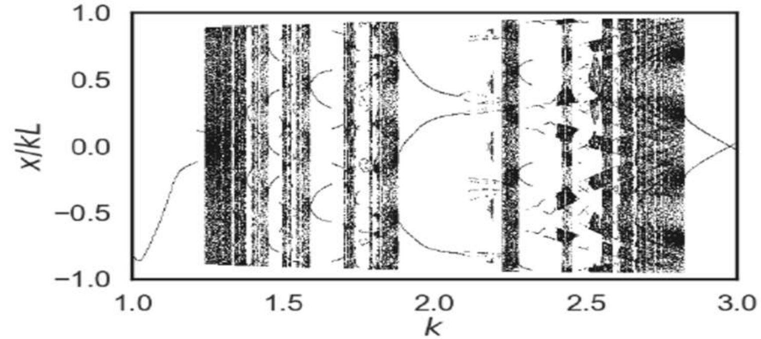


Fig. 1.15 An orbit diagram for (1.11). The horizontal and vertical axes are k and x/kL, respectively.

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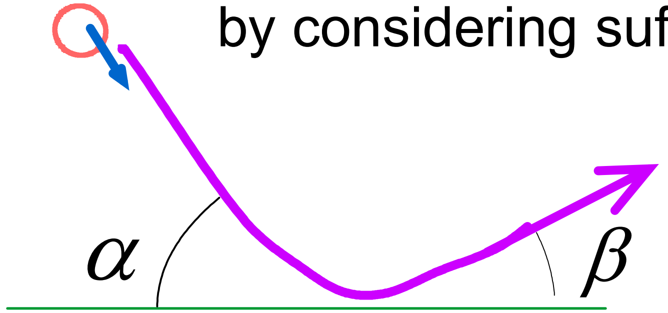
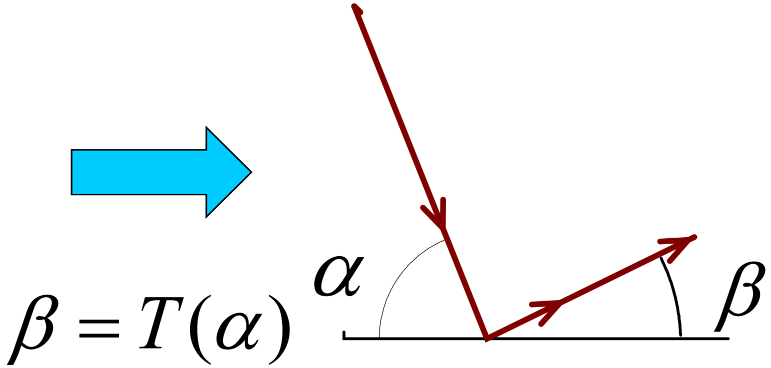
Limiting problem

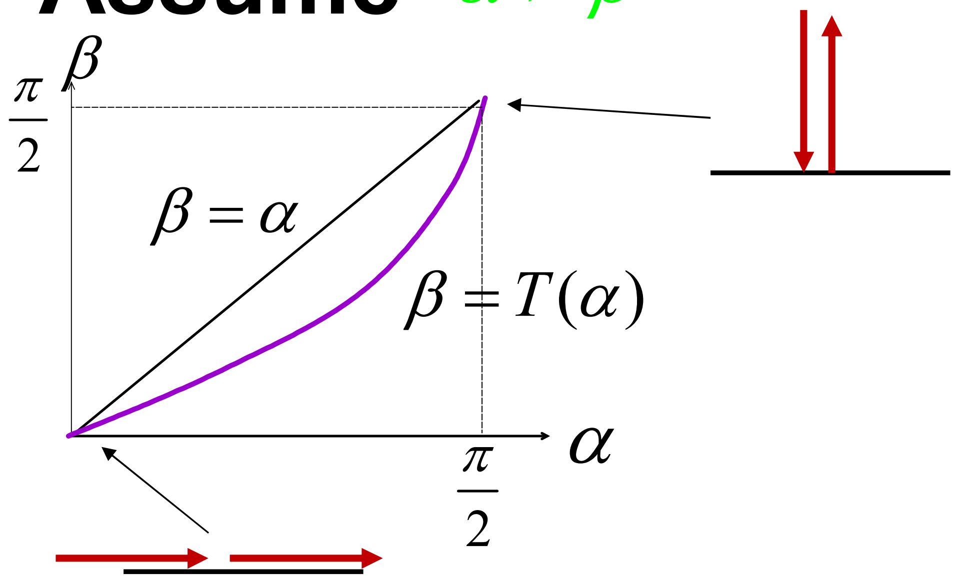
by considering sufficiently large region

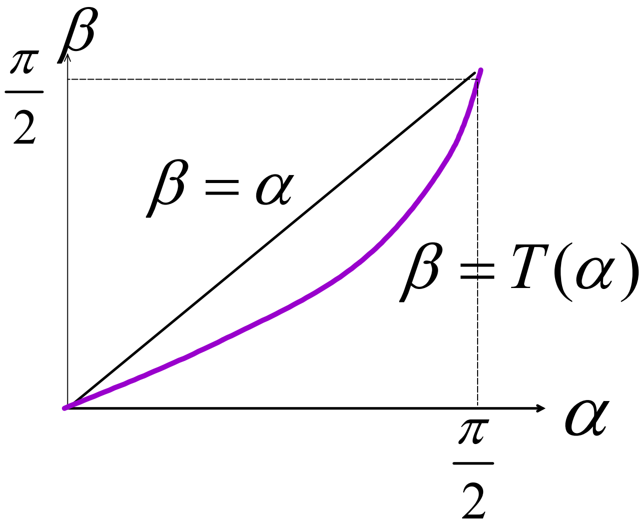


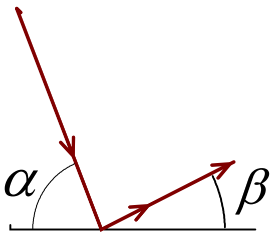


Discrete time model

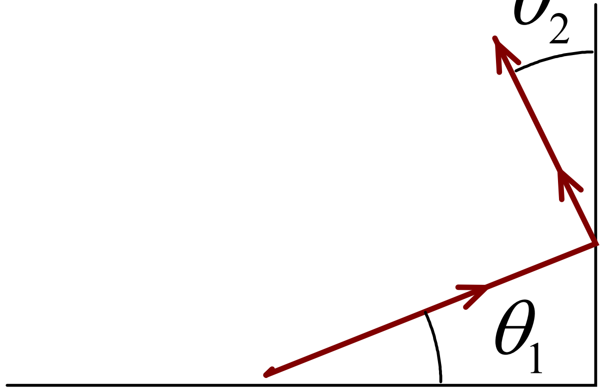
Assume $\alpha > \beta$

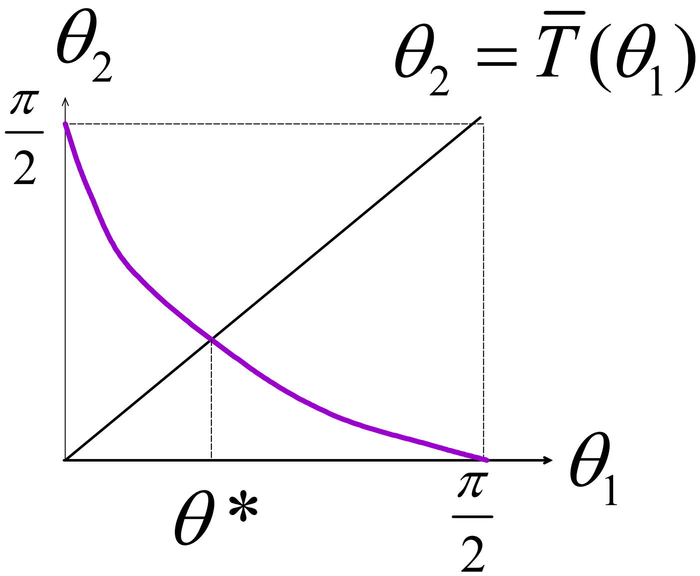




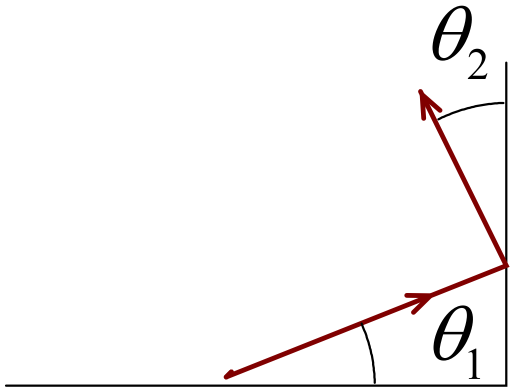


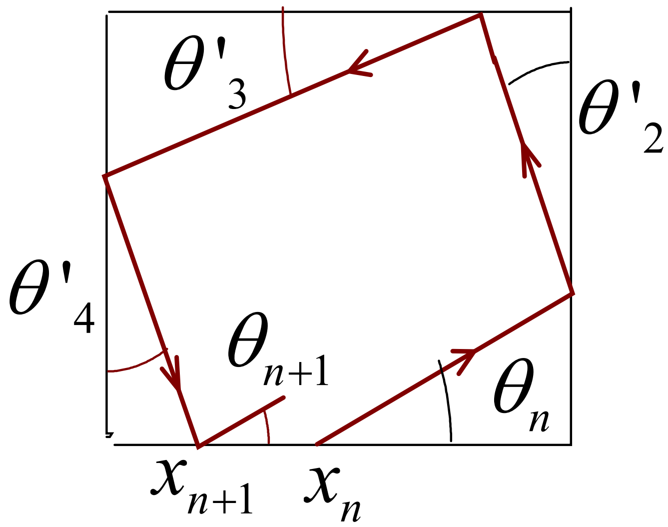
$$\theta_2 = \bar{T}(\theta_1) = T(\pi/2 - \theta_1)$$



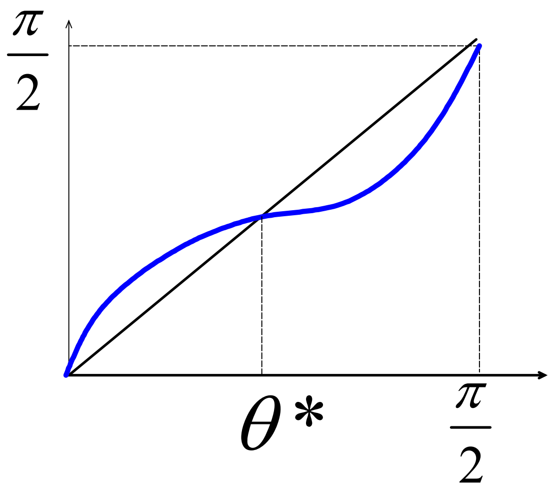


$$\theta_2 = \bar{T}(\theta_1) = T(\pi/2 - \theta_1)$$





$$\theta_{n+1} = \bar{T}^4(\theta_n)$$



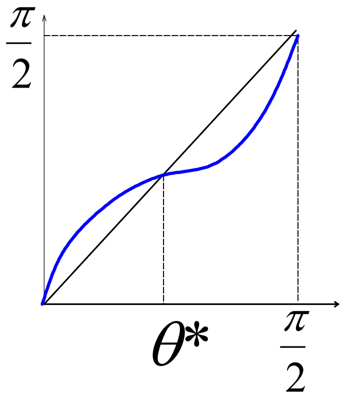
Numerically true (Mimura et.al.)

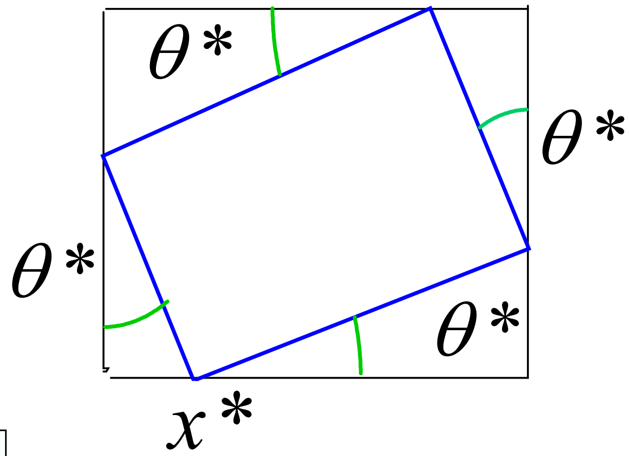
$$\begin{cases} \theta_{n+1} = \bar{T}^4(\theta_n) \\ x_{n+1} = G(\theta_n, x_n) \end{cases}$$

θ^*; stable

$$\theta_n \to \theta^* \ (n \to \infty)$$

$$\begin{cases} \theta_{n+1} = \bar{T}^4(\theta_n) \\ x_{n+1} = G(\theta_n, x_n) \end{cases}$$





If θ^* is stable

Prop.

If $\alpha > \beta = T(\alpha)$, then for θ near θ^*,

$$|G(\theta, x) - G(\theta, y)| \leq q\,|x - y|$$

holds for $0 < q < 1$.

$$\theta^* = \bar{T}^4(\theta^*),$$

⇒ $\theta_n \to \theta^*,\ x_n \to {}^{\exists}x^*\ (n \to \infty)$ $\quad x^* = G(\theta^*, x^*)$

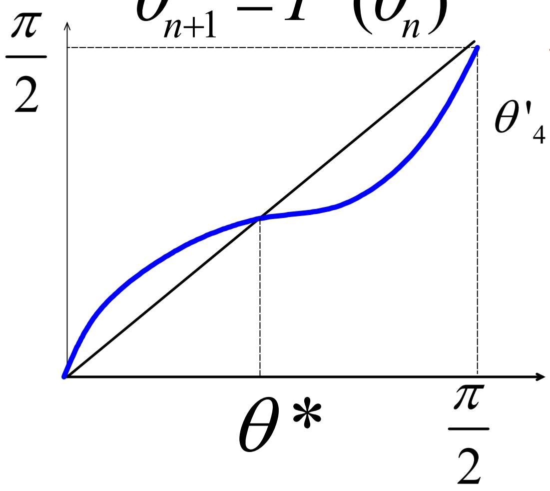
⇒ **Unique existence of stable limit cycle**

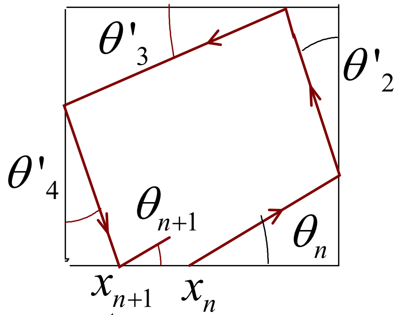
In Discrete time model

Remained problems

Discrete time model

$$\theta_{n+1} = \overline{T}^4(\theta_n)$$

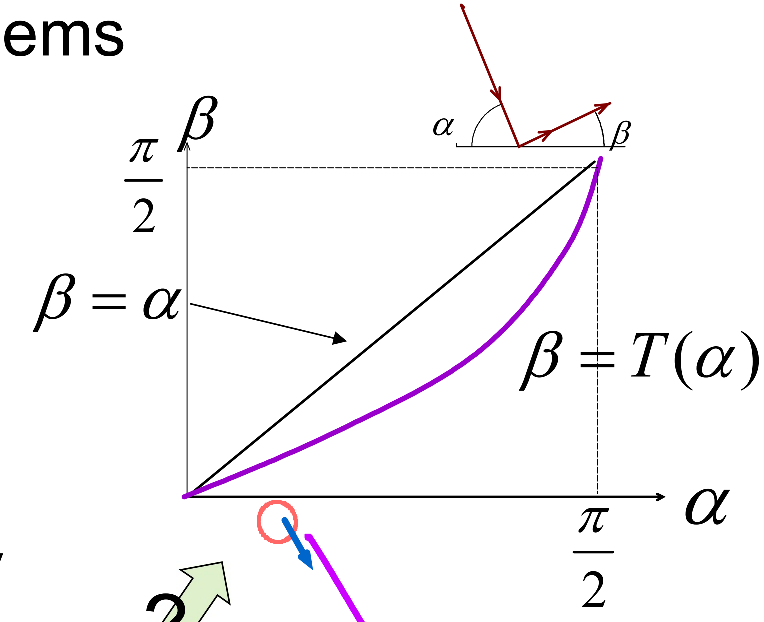




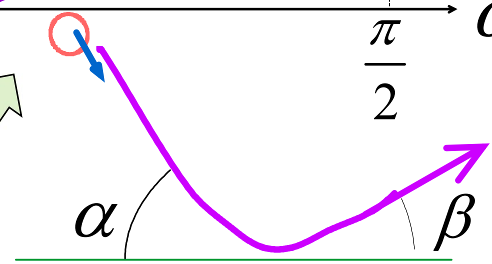
?

$\alpha > \beta$

unsatisfactory



?



Numerically true (Mimura et.al.)

$$\overline{T}(\theta) := T(\pi/2 - \theta)$$

ODE
particle model

$$\begin{cases} \dot{p} = \varsigma, \\ \dot{q} = \xi + \dfrac{M_0}{\sqrt{2q}} e^{-2\alpha q}, \\ \dot{\varsigma} = -(M_1 |\varsigma|^2 - M_2 \eta)\varsigma, \\ \dot{\xi} = -(M_1 |\varsigma|^2 - M_2 \eta)\xi + \dfrac{M'_0}{\sqrt{2q}} e^{-2\alpha q}, \end{cases}$$

MB model

$$\begin{cases} \dfrac{dP}{dt} = \displaystyle\int_{\partial\Omega_c} \gamma(u)\boldsymbol{n} dl, \\ u_t = \Delta u - u + f(\boldsymbol{x}, P), \end{cases}$$

15. Ei, Masayasu Mimura and T. Miyaji, Reflection of a self-propelling rigid disk from a boundary, DCDS-A 2021, 14(3): 803-817
Special issue on recent topics in material, computer and life sciences. doi: 10.3934/dcdss.2020229

Aim to complete this book !

Tomoyuki Miyaji, Shin-Ichiro Ei
Masayasu Mimura

A Billiard Problem in Nonlinear Dissipative Systems

– Monograph –

August 9, 2021

springer

preface

Classical billiard problem has been extensively studied in the community of mathematics. However, this book is concerned with billiard motion of a self-propelling disk in a nonlinear dissipative system on a rectangular pool. As an example of self-propelling disk, a camphor disk floating on water vessel is well-known in nonlinear science. The laboratory experiment demonstrates that, unlike classical billiards, it has the following properties:

(i) It reflects without collision at the boundary, and
(ii) the angle of reflection is greater than that of incidence, which is totally different from elastic reflection.

Relation between real phenomena and models

Real phenomena

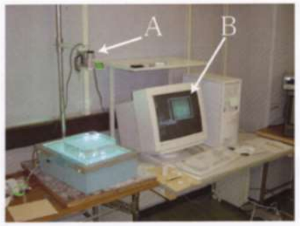


Fig. 1.1 The experimental facility by Kanda[4]. The experiment was recorded by the video camera at A and monitored by the display at B to produce a movie.

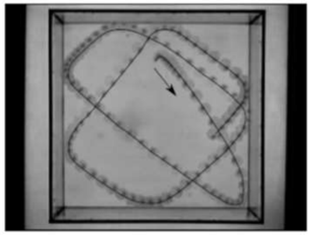
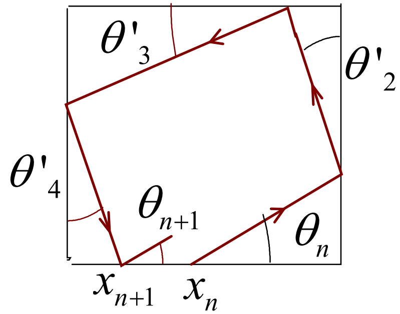


Fig. 1.2 A trajectory of a camphor disk of an experiment by [4].

more realistic but complicated models
(can not be analyzed, many black boxes)

+

Discrete time model

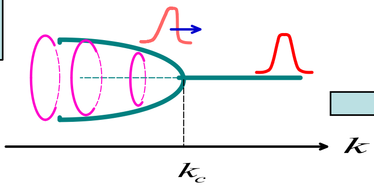


?

MB model

$$\begin{cases} \dfrac{dP}{dt} = \displaystyle\int_{\partial\Omega_c} \gamma(u)\boldsymbol{n}dl, \\ u_t = \Delta u - u + f(\boldsymbol{x}, P), \end{cases}$$

check experimentally



drift bifurcation

ODE
particle model

$$\begin{cases} \dot{p} = \varsigma, \\ \dot{q} = \xi + \dfrac{M_0}{\sqrt{2q}} e^{-2\alpha q}, \\ \dot{\varsigma} = -(M_1 |\varsigma|^2 - M_2\eta)\varsigma, \\ \dot{\xi} = -(M_1 |\varsigma|^2 - M_2\eta)\xi + \dfrac{M'_0}{\sqrt{2q}} e^{-2\alpha q}, \end{cases}$$

Synchronization of theory and experiment

In order to reproduce real phenomena truly

↓

sophistication of model equations:
many variables, complicated nonlinearity, etc

What are checked and measured in experiments ?

Physical phenomena: considerably accurate
Chemical phenomena: considerably accurate for low molecular compounds
Biological phenomena: qualitative properties e.g. monotonicity, on-off effect

Biological systems or systems close to biology

↓

Many unknown factors: many black boxes, no explicit nonlinearities,
unknown number of necessary variables, etc

↓

Necessity of modeling and analysis without any specialization for biological parts

models with black boxes

$$\begin{cases} \dfrac{dP}{dt} = \displaystyle\int_{\partial\Omega_c} \gamma(u)\boldsymbol{n}dl, \\ u_t = \Delta u - u + f(\boldsymbol{x}, P), \end{cases}$$

u ; density of camphor expanding to water

generalization with black boxes

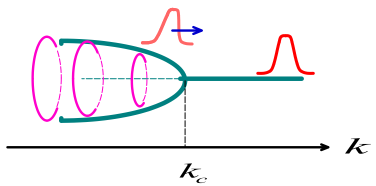
$$\begin{cases} \dfrac{dP}{dt} = \displaystyle\int_{\partial\Omega_c} \gamma_1(U)\boldsymbol{n}dl, \\ U_t = D\Delta U + F_0(U) + F_1(\boldsymbol{x}, P), \end{cases}$$

Rely on parts according to physical law

Corresponding to high molecular parts:
Generalize to vector values and treat as black box

$$u \in \boldsymbol{R} \Rightarrow U \in \boldsymbol{R}^N,\ D := diag\{d_1, \cdots, d_N\},\ d_j > 0$$

+



drift bifurcation

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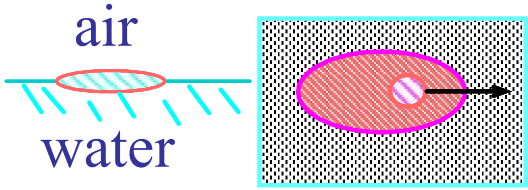
Interaction of Non-radial Camphor tips

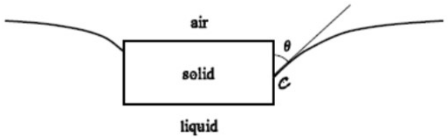
Shin-Ichiro Ei
Hokkaido University
Sapporo, Japan

Partially Joint work with
Nagayama, Kitahata, Koyano
Supported by JST, CREST JPMJCR14D3, Japan

Camphor tip on water surface

樟脳 (防虫剤)





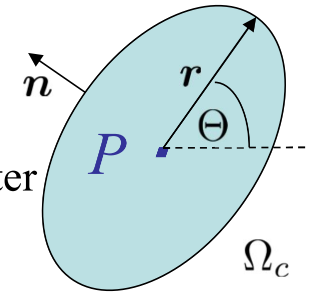
Iida, Kitahata, Nagayama 13

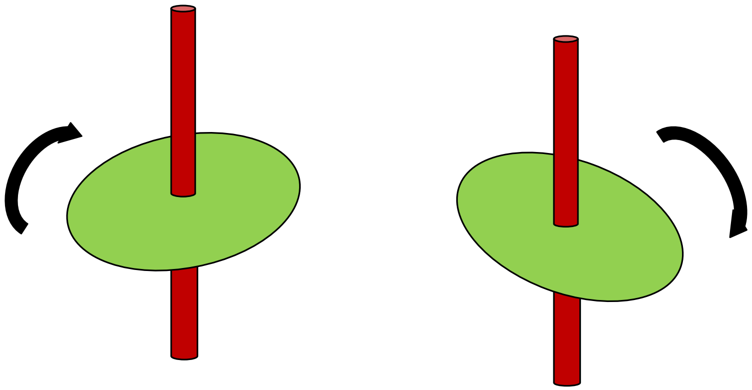
$$\gamma(u) = \frac{\beta^n \gamma_0}{\beta^n + u^n} + \gamma_1$$

$$f(\boldsymbol{x}, \boldsymbol{x}_c, \theta_c) = \begin{cases} f_0, & \boldsymbol{x} \in \overline{\Omega}_c, \\ 0, & \boldsymbol{x} \in \Omega \backslash \overline{\Omega}_c, \end{cases}$$

$$\begin{cases} \dfrac{dP}{dt} &= \displaystyle\int_{\partial\Omega_c} \gamma_1(u)\boldsymbol{n}dl, \\ \dfrac{d\Theta}{dt} &= \displaystyle\int_{\partial\Omega_c} \gamma_2(u)(\boldsymbol{r} \times \boldsymbol{n})dl, \\ u_t &= \Delta u - u + f(\boldsymbol{x}, P, \Theta), \end{cases}$$

u ; density of camphor expanding to water





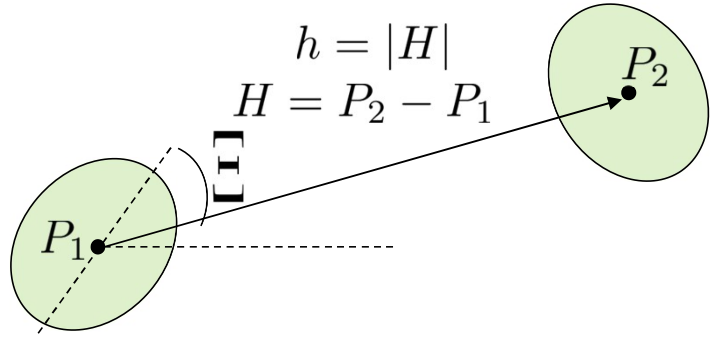
Fix centers $\Longrightarrow$ $\gamma_1 = 0$

Motion of Interaction

E. Nagayama, Kitahata, Koyano 2018

$$\frac{d\Xi}{dt} = \frac{\varepsilon}{\sqrt{h}} e^{-\alpha h} N_m \sin m\Xi$$

$$\Gamma_\varepsilon := \{(r_0 + \varepsilon \cos m\theta)\boldsymbol{e}(\theta)\}$$



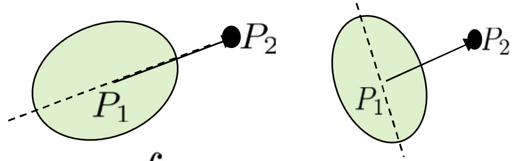
$N_m < 0, \; N_m > 0$

$$\Xi \to 0, \; \pm\frac{\pi}{m}$$

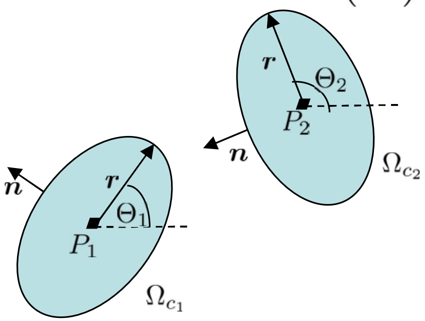
Ex. *m* = 2 (ellipse)

$$\Xi \to 0, \; \pm\frac{\pi}{2}$$

Apsis or minor axis



Note: $\Theta = \Theta_2 - \Theta_1$ appears in $O(\varepsilon^2)$



<u>Prop.</u>

$$\begin{cases} \dfrac{dP}{dt} &= \displaystyle\int_{\partial\Omega_c} \gamma_1(u)\boldsymbol{n}\,dl, \\ \dfrac{d\Theta}{dt} &= \displaystyle\int_{\partial\Omega_c} \gamma_2(u)(\boldsymbol{r}\times\boldsymbol{n})\,dl, \\ u_t &= \Delta u - u + f(\boldsymbol{x}, P, \Theta), \end{cases}$$

then

$$N_m = 2\pi I_m(\alpha r_0) > 0, \; \Xi \to \frac{\pi}{m}$$

*m*次の第1種変形ベッセル関数

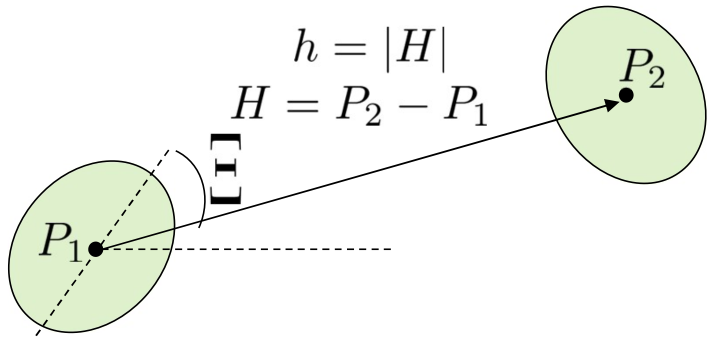
Motion of interaction : Mode 2

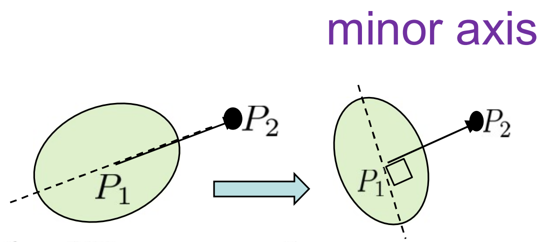
$$\frac{d\Xi}{dt} = \frac{\varepsilon}{\sqrt{h}} e^{-\alpha h} N_2 \sin 2\Xi$$

m = 2
(ellipse)

$$N_2 > 0, \ \Xi \to \frac{\pi}{2}$$

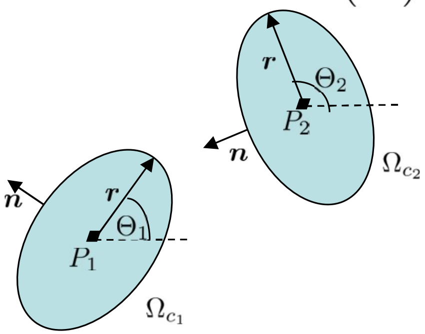
$$\Gamma_\varepsilon := \{(r_0 + \varepsilon \cos 2\theta)\boldsymbol{e}(\theta)\}$$

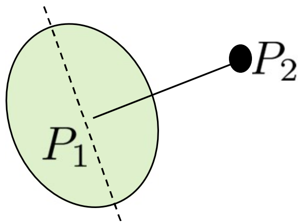




$$\begin{cases} \dfrac{dP}{dt} = \displaystyle\int_{\partial\Omega_c} \gamma_1(u)\boldsymbol{n}\,dl, \\ \dfrac{d\Theta}{dt} = \displaystyle\int_{\partial\Omega_c} \gamma_2(u)(\boldsymbol{r}\times\boldsymbol{n})\,dl, \\ u_t = \Delta u - u + f(\boldsymbol{x}, P, \Theta), \end{cases}$$

Note: $\Theta = \Theta_2 - \Theta_1$
appears in $O(\varepsilon^2)$





Motion of interaction : Mode 3

$$\frac{d\Xi}{dt} = \frac{\varepsilon}{\sqrt{h}} e^{-\alpha h} N_3 \sin 3\Xi$$

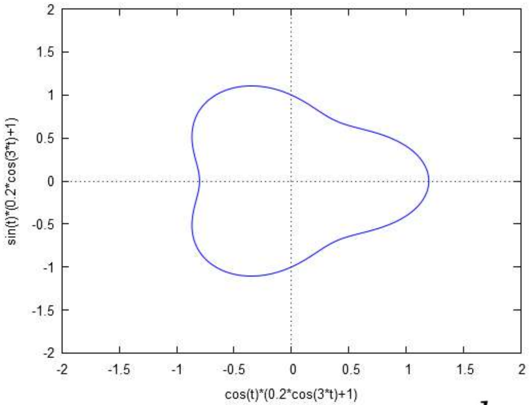
→

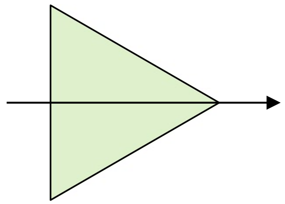
$$N_3 > 0,\ \Xi \to \frac{\pi}{3}$$

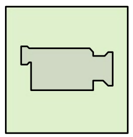
$$\Gamma_\varepsilon := \{(r_0 + \varepsilon \cos 3\theta)\boldsymbol{e}(\theta)\}$$

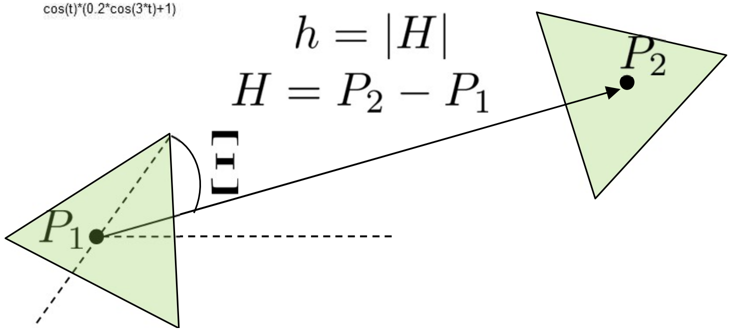
m **= 3**

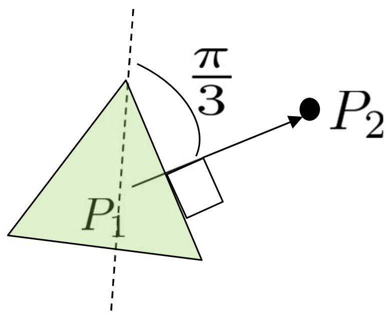
$$\begin{cases} \dfrac{dP}{dt} &= \displaystyle\int_{\partial\Omega_c} \gamma_1(u)\boldsymbol{n}\,dl, \\ \dfrac{d\Theta}{dt} &= \displaystyle\int_{\partial\Omega_c} \gamma_2(u)(\boldsymbol{r}\times\boldsymbol{n})\,dl, \\ u_t &= \Delta u - u + f(\boldsymbol{x}, P, \Theta), \end{cases}$$





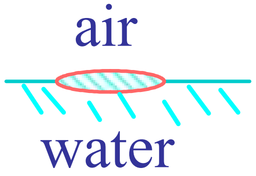


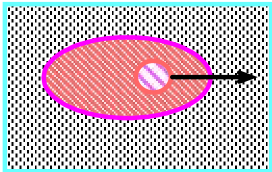


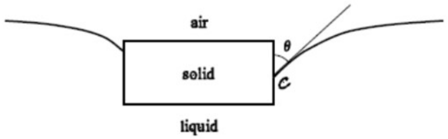


Camphor tip on water surface

樟脳 (防虫剤)







Iida, Kitahata, Nagayama 13

$$\gamma(u) = \frac{\beta^n \gamma_0}{\beta^n + u^n} + \gamma_1$$

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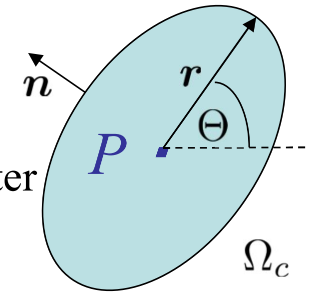


u ; density of camphor expanding to water

$$\begin{cases} \dfrac{dP}{dt} = \displaystyle\int_{\partial\Omega_c} \gamma_1(U)\boldsymbol{n}\,dl, \\ \dfrac{d\Theta}{dt} = \displaystyle\int_{\partial\Omega_c} \gamma_2(U)(\boldsymbol{r}\times\boldsymbol{n})\,dl, \\ U_t = D\Delta U + F_0(U) + F_1(\boldsymbol{x}, P, \Theta), \end{cases}$$

Rely on parts according to physical law

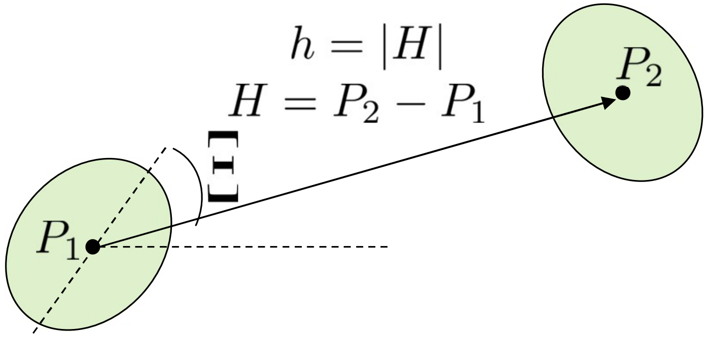
Corresponding to high molecular parts:
Generalize to vector values and treat as black box

$$u \in \boldsymbol{R} \Rightarrow U \in \boldsymbol{R}^N,\ D := diag\{d_1, \cdots, d_N\},\ d_j > 0$$

Only assume the existence and stability of stationary solution when ***F*₁** is given.

Motion of Interaction

$$\Gamma_\varepsilon := \{(r_0 + \varepsilon \cos m\theta)\boldsymbol{e}(\theta)\}$$



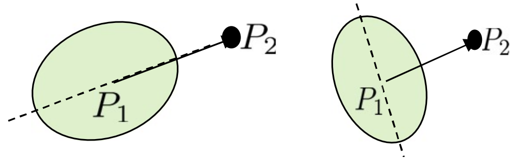
Ex. *m* = 2 (ellipse)

$N_m < 0,\ N_m > 0$

$$\Xi \to 0,\ \pm\frac{\pi}{m}$$

$$\Xi \to 0,\ \pm\frac{\pi}{2}$$

Apsis or minor axis

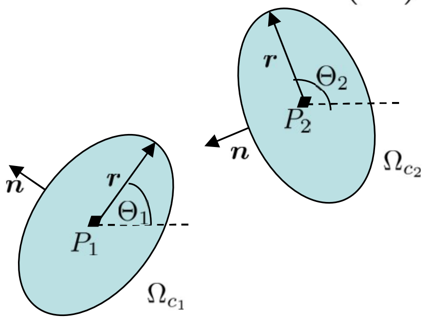


Universal equation

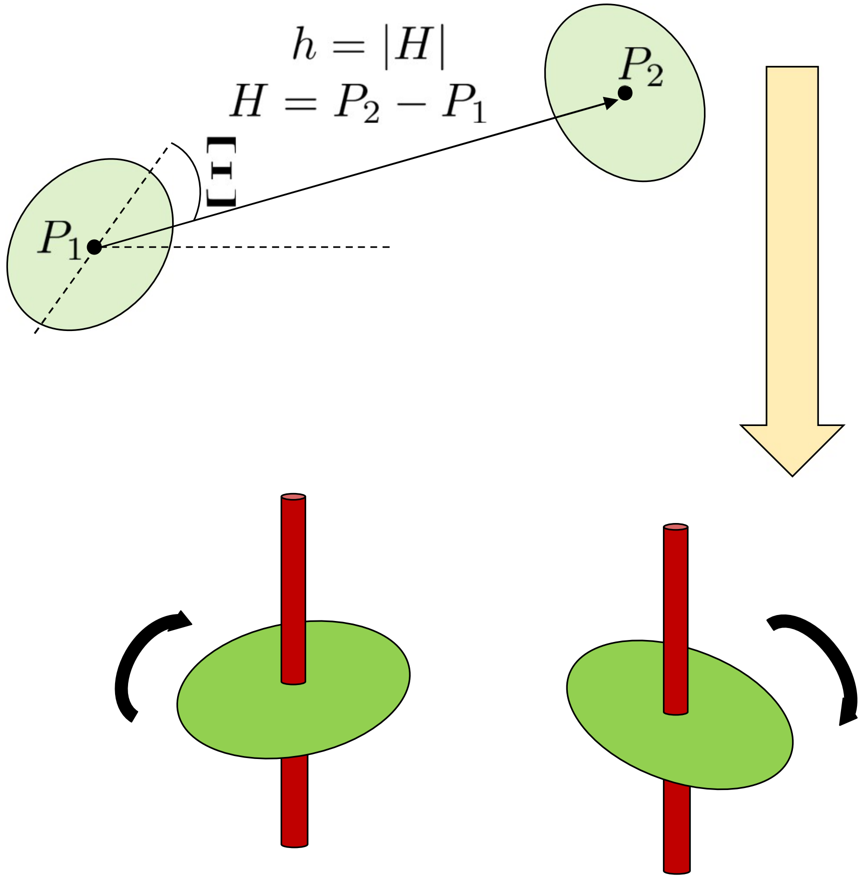
$$\frac{d\Xi}{dt} = \frac{\varepsilon}{\sqrt{h}} e^{-\alpha h} N_m \sin m\Xi$$

Nm is determined by experiments

Note: $\Theta = \Theta_2 - \Theta_1$ appears in $O(\varepsilon^2)$

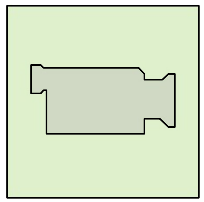


Experiment



Fix centers ⟹ $\gamma_1 = 0$

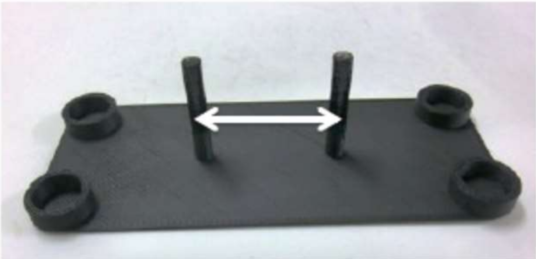
experiment



Experiment

実験内容

＊楕円形樟脳粒を水に1つ浮かべて
自発回転しないことを確認



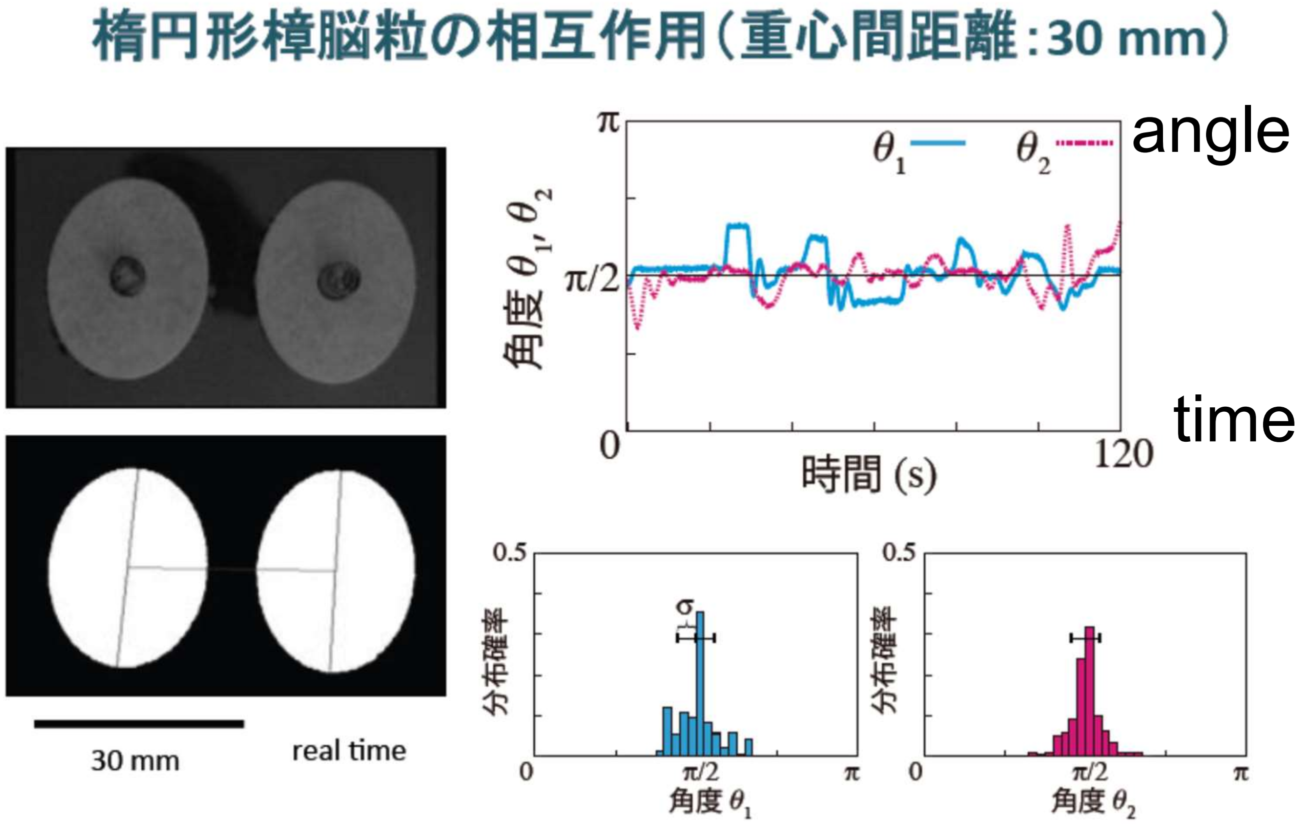
＊楕円形樟脳粒を純水に2つ浮かべて
向きの時間変化を観察

樟脳ろ紙同士の距離の変更は
軸の位置を変化させることで実現

楕円形樟脳粒同士の重心間距離を変化させて、
向きの揃う様子を観察

楕円形樟脳粒を適当な向きに固定しておいて、
離したあとの様子を観察（安定性のチェック）

Interaction of two elliptic camphors



summary

- Model for two camphor particles with deformation is considered.

- The equation describing the motion of angles are derived
  for general reaction terms.

- In the case of small deformation from radial symmetry,
  the equation is explicitly derived, even with black boxes.

Thank you for your attention

with a lot of memories of Mimura sensei

"Convergence, concentration and critical mass phenomena for a model of cell migration with signal production on the boundary"

Philippe Souplet (Université Sorbonne Paris Nord, France)
(joint work with Nicolas MEUNIER (Université d'Evry-Val d'Essonne))

We consider a model of cell migration with signal production on the boundary. It consists in a diffusion equation with nonlinear nonlocal advection, complemented by a no-flux condition ensuring mass conservation.

For nonlinearities with polynomial growth, we first develop a local existence-uniqueness theory in optimal L^p spaces. With help of this tool, we next obtain the following results on the global behavior of solution:

- For small initial data, we have exponential convergence towards a constant.
- If, and only if, the growth of the nonlinearity is at least quadratic, we have concentration, i.e. finite time blowup, for large initial data.
- In the critical case of a quadratic nonlinearity, we observe a critical mass phenomenon *in any space dimension* (denoting by M the mass of the L^1 initial data):
  - for $M \leq 1$, the solution is global and bounded;
  - for $M > 1$, there exist initial data leading to finite time concentration.

This critical mass phenomenon is reminiscent of the well-known situation for the 2D Keller-Segel system. The global existence proof is delicate, based on a control of the solution by means of an entropy functional, via an ε-regularity type result.

- Finally we give some partial results on the localization and final profile of the boundary concentration and on the blowup rate.

CONVERGENCE, CONCENTRATION AND CRITICAL MASS PHENOMENA FOR A MODEL OF CELL MIGRATION WITH SIGNAL PRODUCTION AT THE BOUNDARY

Philippe Souplet

LAGA, Université Sorbonne Paris Nord

Joint work with Nicolas Meunier, Université Évry Val d'Essonne, France

ICMMA – Meiji University, November 2023

In memory of Professor Masayasu Mimura

BIOLOGICAL BACKGROUND: CELL MIGRATION

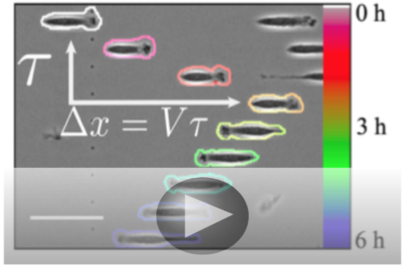
- Cell migration: fundamental process in physiological and pathological functions (immune response, morphogenesis, cancer metastasis, etc)

BIOLOGICAL BACKGROUND: CELL MIGRATION

- Cell migration: fundamental process in physiological and pathological functions (immune response, morphogenesis, cancer metastasis, etc)
- Cell migration obeys general principle:
  Strong correlation between direction of trajectories and velocity of cells
  (fastest cells have more directional migration)

The first World Cell Race [M. Piel et al. Current Biology. 2013]

- Race track: $4\mu m \times 12\mu m$ fibronectin lines.
- 54 different cell types from various animal and tissues, provided by 47 laboratories. Genotypically WT, transformed, or genetically engineered.
- Total of 7 000 cells. About 130 cells per cell type.
- Winner: human embryonic mesenchymal stem cell, $5.2\mu m/min$.
- Correlation between polarization and instantaneous cell speed. Seems to be universal.



CELL MIGRATION MECHANISMS

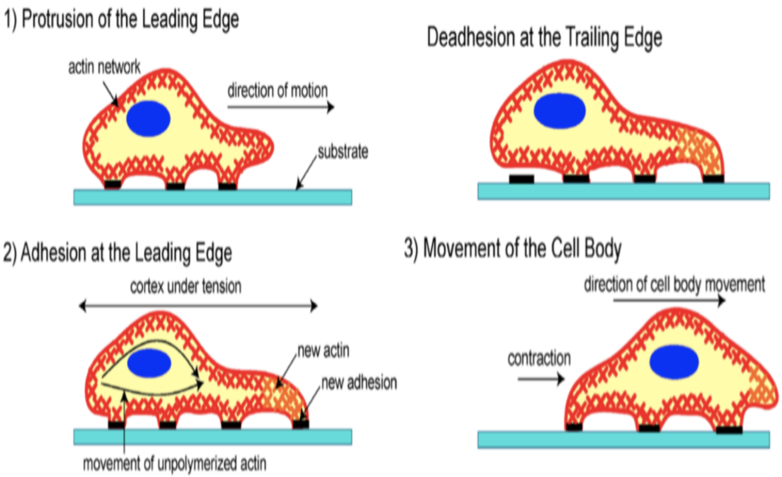
• Actin filaments: essential components of cytoskeleton of eukaryotic cells

• Polymerize and grow at one end, depolymerize and retract at other end (near cell membrane)

• Retrograde actin flow

Large, fast actin flows enhance cell polarity, hence cell persistence time

• Actin flows as a result of advection of polarity signals (= molecules involved in regulation of cytoskeleton activity)



MODEL

$u = u(t,x)$ concentration of solute located in cell
(cytoplasmic protein controlling active force-generation/adhesion machinery of cell)

$$(P) \qquad \begin{cases} u_t = \nabla \cdot \big(\nabla u - A(t)u\big), & t > 0,\ x \in \Omega, \\ 0 = \big(\nabla u - A(t)u\big) \cdot \nu, & t > 0,\ x \in \partial\Omega, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases}$$

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  - molecules advected by actin fluxes and activated at cell membrane
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  - higher concentration gradient across cell $\Longrightarrow$ faster actin flow

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- References

[Calvez, Meunier, Voituriez, CRAS 2010]
[Calvez, Hawkins, Meunier, Voituriez, SIAP 2012]
[P. Maiuri, R. Voituriez et al., Cell 2015]
[Lavi-Meunier-Voituriez-Casademunt, Phys. Rev. 2020]

MOTIVATIONS AND AIMS

- Investigate influence of nonlinearity f on global or nonglobal solvability
- Can concentration occur (on $\partial\Omega$) ? finite time blow-up ?
- Understand global and/or blow-up behavior
- Local well-posedness for rough (L^q) initial data (key tool !)

Typical nonlinearities

$$f(u) = u^p, \quad p \geq 1$$

$$f(u) \sim u^p \ \text{ at } \infty, \quad 0 < p < 1 \quad (f \in C^1,\ f \geq 0)$$

[• Traveling waves for nonlinearities with saturation $f(u) = u/(C + u)$ – ongoing project]

BASIC FEATURES

1. Conservation of mass (total amount of solute).

No-flux condition $\Longrightarrow$

$$\int_\Omega c_0(t,x)\,dx = M := \int_\Omega c_0(x)\,dx$$

$\to$ special role of space L^1

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2. Order of nonlinearity

Divergence formula $\Longrightarrow$

$$A(t) := \int_{\partial\Omega} f(u)\nu d\sigma = \int_\Omega f'(u)\nabla u\,dx$$

Eqn. becomes:

$$\boxed{u_t - \Delta u = p\nabla u \cdot \int_\Omega u^{p-1}\nabla u\,dx}$$

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3. Scale invariance (e.g. half-space case $\Omega = \{x \in \mathbb{R}^n;\ x_n > 0\}$)

$$u \text{ solution} \quad \Longrightarrow \quad u_\lambda(x,t) = \lambda^{n/p} u(\lambda x, \lambda^2 t) \ \text{ also solution} \quad (\lambda > 0)$$

Invariant L^q-norm for $q = p$:

$$\|u_\lambda(\cdot,0)\|_{L^p} \equiv \|u(\cdot,0)\|_{L^p}, \quad \lambda > 0$$

RELATED NONLOCAL PROBLEMS

- Nonlocal Neumann problems with zero order nonlinearities

$$\begin{cases} u_t - \Delta u = f_1(u)\big(\int_\Omega f_2(u)\,dx\big), & t > 0,\ x \in \Omega \\ u_\nu = g_1(u)\big(\int_\Omega g_2(u)\,dx\big), & t > 0,\ x \in \partial\Omega \end{cases}$$

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• Equations with nonlocal gradient terms (and homogenous boundary conditions)

$$u_t - u^m\Delta u = u^p \int_\Omega |\nabla u|^2\,dx$$

$m=p=1$: model of replicator dynamics

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- Nonlocal problem with mass control:

$$u_t - \Delta u = \frac{\lambda e^u}{\int_\Omega e^u\,dx}$$

(related with Keller-Segel system)

[Wolansky 97, Kavallaris-Suzuki 07, 18]

OUR PROBLEM

Conservative form

$$\begin{cases} u_t = \nabla \cdot \big(\nabla u - A(t)u\big), & t > 0,\ x \in \Omega, \\ 0 = \big(\nabla u - A(t)u\big) \cdot \nu, & t > 0,\ x \in \partial\Omega, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases}$$

Convective-diffusive form

$$(P) \qquad \begin{cases} u_t - \Delta u = -A(t) \cdot \nabla u, & t > 0,\ x \in \Omega, \\ u_\nu = (A(t) \cdot \nu)u, & t > 0,\ x \in \partial\Omega, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases}$$

$$A(t) = \int_{\partial\Omega} f(u)\nu d\sigma = \int_{\Omega} f'(u)\nabla u \, dx$$

$$f(u) = u^p \quad (p > 0)$$

LOCAL WELL-POSEDNESS FOR ROUGH DATA

Standard theory $\Longrightarrow$ local well-posedness for smooth initial data (e.g. $C^1(\overline{\Omega})$).

Theorem 1.

Let $p > 0$, $q \geq \max(p, 1)$, $u_0 \in L^q(\Omega)$.

(i) Problem (P) admits a unique maximal classical positive solution

$$u \in C^{2,1}(\overline{\Omega} \times (0, T^*)) \cap C([0, T^*); L^q(\Omega)).$$

(ii) If $q > p$ and $T^* < \infty$, then $\lim_{t \to T^*} \|u(t)\|_q = \infty$.

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Remarks

- Critical role played by L^p norm:
  - continuation property (ii) fails for $1 \leq q < p$ or $q = p = 1$,
  - for $p = 1$, entropy blows up whenever $T^* < \infty$, namely

$$\lim_{t \to T^*} \int_\Omega (u \log u)(t)\, dx = \infty$$

- Condition $q \geq p$ in Theorem 1 also natural in view of scaling properties
- Open problem whether local existence may fail when $1 \leq q < p$
- L^1 scaling critical case belongs to local existence range

($\neq$ Fujita equation ! $u_t - \Delta u = u^p$, $p = 1 + \frac{2}{n}$, $q = 1$)

Theorem 2.

Let $f(u) = u$, $u_0 \in L^1(\Omega)$ *and set* $M = \|u_0\|_1$.

(i) If $M \leq 1$, *then* $T^* = \infty$ *and* $\sup_{t \geq 1} \|u(t)\|_\infty < \infty$

(ii) If $M > 1$ *and* Ω *is a cylinder, then there exist* u_0 *such that* $T^* < \infty$.

CRITICAL MASS PHENOMENON FOR $p = 1$

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Remarks

- Critical mass phenomenon with sharp threshold $M = 1$.
- Reminiscent of well-known situation for $2d$ Keller-Segel system.

Main difference:
- critical mass phenomenon is dimension-independent
- solutions with critical mass remain bounded, unlike critical mass case of $2d$ KS

$$\bar{u}_0 := \frac{1}{|\Omega|} \int_\Omega u_0 \, dx \qquad \text{(average of } u_0\text{)}$$

Theorem 3. *Let $u_0 \in L^1$.*

(i) If $0 < p < 1$, then $T^* = \infty$ and $\sup_{t \geq 1} \|u(t)\|_\infty < \infty$

(ii) If $p > 1$ and $M > 0$, then there exists u_0 such that $T^* = \infty$ and $\|u_0\|_1 = M$

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Sign of nonlinearity is important ! :

Theorem 4. *Let $f(u) = -u^m$ with $m \geq 1$ and $u_0 \in L^m(\Omega)$. Then $T^* = \infty$ and $\sup_{t\geq 1}\|u(t)\|_\infty < \infty$.*

Theorem 5.

(i) Let $p = 1$ *and* $u_0 \in L^1$.

- *If* $M < 1$, *then*

$$\lim_{t\to\infty} \|u(t) - \bar{u}_0\|_\infty = 0, \qquad \bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0\,dx.$$

- *If* $M = 1$, *there exist nonconstant steady states.*

(ii) Let $p > 0$ *and* $u_0 \in L^q$, $q = \max(p, 1)$.

- *If* $\|u_0\|_q \ll 1$ *then* $T^* = \infty$ *and*

$$\lim_{t\to\infty} \|u(t) - \bar{u}_0\|_\infty \le Ce^{-\lambda t}.$$

- *There also exist nonconstant steady states.*

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Remark. Previous result [Calvez, Hawkins, Meunier, Voituriez, SIAP 2012]
$u_t = (u_x - u(0,t)u)_x$ on half-line $I = (0, \infty)$, with zero flux condition at $x = 0$
Existence of a global, suitable weak solution
for $u_0 \in L^1(I, (1 + x)dx)$, $u_0 \log u_0 \in L^1(I)$, $M \leq 1$

ASYMPTOTIC BEHAVIOR OF BLOWUP SOLUTIONS

- Cylinder $\Omega = (0, L) \times B'_R \subset \mathbb{R}^n$ if $n \geq 2$, or $\Omega = (0, L)$ if $n = 1$
- Initial data $u_0 \in C^1(\overline{\Omega})$, axisymmetric with respect to e_1 (if $n \geq 2$), $\partial_{x_1} u_0 \leq\not\equiv 0$

Theorem 6.

(i) Assume $n \geq 2$. Then

$$u(x,t) \leq C x_1^{-1} \quad \text{in } \Omega \times (0, T^*).$$

In particular, if $T^* < \infty$ then blow-up set $\subset \partial\Omega \cap \{x_1 = 0\}$.

(ii) Assume $n = 1$. Then

$$u(x,t) \leq (px)^{-1/p} + C \quad \text{in } (0, L] \times (0, T^*).$$

In particular, if $T^* < \infty$ then blow-up set $= \{0\}$

(iii) Lower blow-up estimate:

$$\|u(t)\|_\infty \geq C(T^* - t)^{-1/2p}, \; 0 < t < T^*$$

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Remark: power $1/p$ is optimal (open question for $n \geq 2$)

Open problems (possibly difficult)

- upper time rate estimate (type I/II ?)
- single point BU in cylinder
- BU for other domains (difficulty: monotonicity)
- attractivity of nonconstant steady-states

IDEAS OF PROOFS: Theorem 1 (local existence for L^q data)

- Approximation by smooth initial data
- Semigroup techniques / smoothing effects via representation formula

$$u(t) = S(t)u_0 + \int_0^t K_\nabla(t-s)[A(s)u(\cdot,s)]\,ds, \quad 0 < t < T,$$

with differentiated semigroup

$$[K_\nabla(t)\psi](x) = \int_\Omega \nabla_y G(x,y,t)\cdot\psi(y)\,dy, \quad \psi \in (L^1(\Omega))^n$$

- Fixed point in fractional Sobolev spaces with time-weighted intermediate norm

$$\sup_{t\in(0,T)} t^{1/2q}\|u_i(t)\|_{1/q,q}$$

- Use interpolation + trace inequality $W^{1/q,q}(\Omega) \subset L^q(\partial\Omega)$

Remark. Critical case $p = q$ (especially $p = q = 1$) more delicate

IDEAS OF PROOFS: Theorem 2 (blowup in a cylinder – case $p = 1$, $M > 1$)

- Assume $u_{0,x_1} \leq 0$. Maximum principle $\Longrightarrow u_{x_1} \leq 0$
- Auxiliary functional: first moment

$$\phi(t) = \int_\Omega x_1 u \, dx \geq 0$$

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- IBP + B.C., mass conservation $\int_\Omega u = M$ and $A(t) = \int_\Omega \nabla u \, dx \Longrightarrow$

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- For $\Omega = (0,1)$: $u(t,0) \ge M, \quad u(t,1) = u(t,1) \int_0^1 2x dx \le \int_0^1 2xu(t,x) dx = 2\phi(t)$

$$\Longrightarrow \phi' \le -(M-1)(M-2\phi(t))$$

But $2\phi(0) = \int_0^1 2xu_0(x)dx < \int_0^1 2xu_0(x^2)dx = \int_0^1 u_0(z)dz = M$

$$\Longrightarrow \phi' \le -\eta < 0 : \quad \text{contradiction}$$

IDEAS OF PROOFS: Theorem 2 (global existence – case $p = 1$, $M \le 1$)

- Entropy function $\phi(t) := \int_\Omega (u \log u + 1)\, dx > 0$.

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$$\phi'(t) = -\int_\Omega u^{-1} |\nabla u|^2\, dx + \left| \int_\Omega \nabla u\, dx \right|^2$$

Cauchy-Schwarz and $M \leq 1 \Longrightarrow$

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- An ε-regularity property:

$$\begin{cases} u_0 = u_0^1 + u_0^2 \\ \|u_0^1\|_1 \ll 1,\ \|u_0^2\|_\infty \le K \end{cases} \Longrightarrow \text{ smoothing effect in } L^\infty \text{ with uniform time } \tau(K) > 0$$

- Entropy bound + ε-regularity $\Longrightarrow$ uniform L^∞ estimate

IDEAS OF PROOFS (continued)

Theorem 3 (global existence – case $p < 1$)

- L^1 norm is scaling supercritical

$\Longrightarrow L^\infty$ smoothing effect from L^1 bound (by local theory)

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Theorem 5 (asymptotic behavior)

• case $p = 1$, $M < 1$. Use entropy as Liapunov functional:

$$\int_1^\infty \int_\Omega u^{-1}|\nabla u|^2 \, dxdt < \infty$$

$\Longrightarrow \omega$ limits are space-independent hence $\equiv \bar{u}_0$ by mass conservation

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Theorem 6 (blow-up space asymptotics – case $n = 1$)

• Auxiliary functional

$$\phi := u_x + u^{p+1} - K_1 u - K_2$$

Maximum principle $\Longrightarrow \phi \leq 0$

• Integrate in $x \Longrightarrow u(x,t) \leq (px)^{-1/p} + C$

CONCLUSIONS

- Optimal results on local well-posedness in L^q spaces (critical exponent $q = p$)
- Optimal results on global solvability (critical exponent $p = 1$)
- Sharp mass threshold phenomenon for $p = 1$ (critical mass $M = 1$)
- Asymptotic convergence to constants for subcritical mass or small data
- Nonconstant steady states
- Partial information on blow-up set, rate and profiles
- **Open problems**

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 - BU for other domains
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- BU for other domains
- attractivity of nonconstant steady-states
- ...

THANK YOU FOR YOUR ATTENTION !

"Front propagation in the presence of obstacles"

Hiroshi Matano (Meiji University, Japan)

In this talk I will discuss the effect of geometric obstacles on the propagation of fronts. Two types of fronts are considered. One is a transition layer in a bistable reaction diffusion equation. The other is a curvature-dependent motion of plane curves. Both types of fronts are closely related. In the first part, I will present my joint work with Henri Berestycki and François Hamel. I will then discuss the curvature-dependent motion of plane curves through an infinite channel with saw-toothed boundaries. For this second topic, I will first recall my joint work with Ken-Ichi Nakamura and Bendong Lou (2006, 2013), which deal with domains with mildly undulating boundaries. I will then discuss my ongoing joint work with Ryunosuke Mori, which deals with domains whose boundaries have steeper bumps. In such domains, a new type of phenomenon, which we call "obstacle-induced propagation", can be observed.

Front Propagation in the presence of obstacles

Hiroshi Matano

（Meiji University)

ICMMA 2023 “Reaction-diffusion systems: from the past to the future”

MIMS, Meiji University, November 2, 2023

Chronology (Prof. Mimura)

- ◆ 1941: born in Takamatsu City.
- ◆ 1965: graduated Kyoto University.
- ◆ 1970-79: Konan University
- ◆ 1976-77: visiting Oxford University
- ◆ 1980: Hiroshima University
  (1993—1999: Univ. Tokyo)
- ◆ 2004: Meiji University

Masayasu Mimura
(Mayan)

Oct 11, 1941 – Apr 8, 2021

advisor: Prof. Masaya Yamaguti

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That was a turning point in my life!

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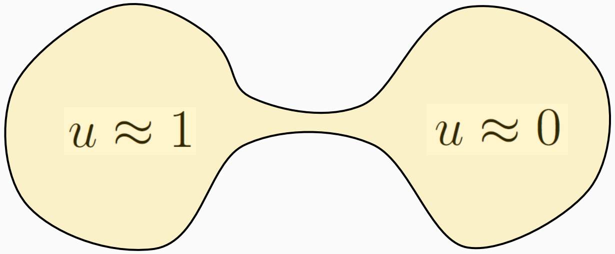

My work on dumbbell-shaped domain was inspired by his question.

Tokyo 2023

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Matano

- ◆ 1975: entered Master's Course at Kyoto University
- ◆ 1979 - 82: Univ. Tokyo
- ◆ 1982 - 88: Hiroshima University (Mimura, Kobayashi, Ei, …)
- ◆ 1988 - 2018: Univ. Tokyo
- ◆ 2018 - 2023: Meiji University

01 May 2005: at Taro Okamoto Museum, Kawasaki, Japan

Outline of the talk

1. Introduction
2. Front propagation through a perforated wall
3. Propagation in a saw-toothed cylinder

   Curvature-dependent motion of interfaces

4. Previous results

   Motion without singularity

5. Main results

   Traveling waves with singularities &
   Obstacle-induced propagation

6. Numerical simulation

Joint work with Ryunosuke Mori (Meiji University)

1. Introduction

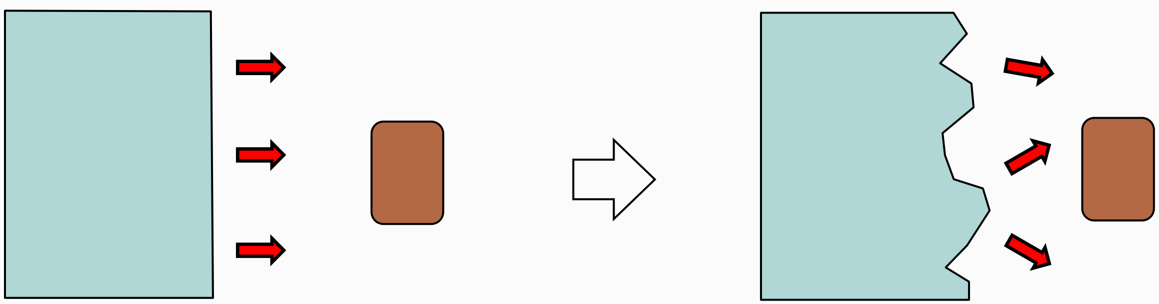
Propagation in the presence of obstacles

Front propagation in the presence of obstacles

Theme 1 — Effect of localized obstacles

$$\frac{\partial u}{\partial t} = \Delta u + f(u) \quad \text{in} \quad \Omega := \mathbb{R}^N \setminus K$$

K : obstacle



planar front

Incidental distubance

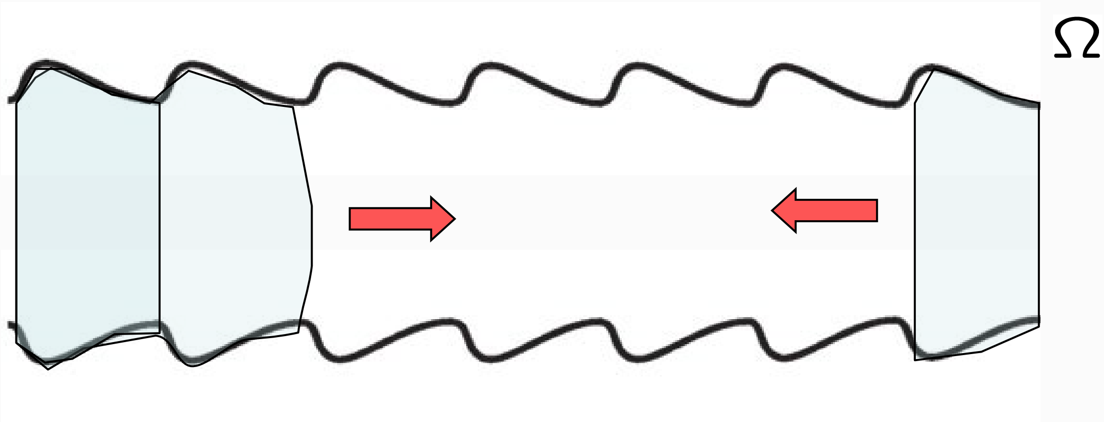
Q What is the long-time effect of this incidental disturbance?

(Case1) K : compact obstacle — Berestycki-Hamel-M. (CPAM 2009)

(Case2) K : perforated wall — subject of the present talk

Theme 2

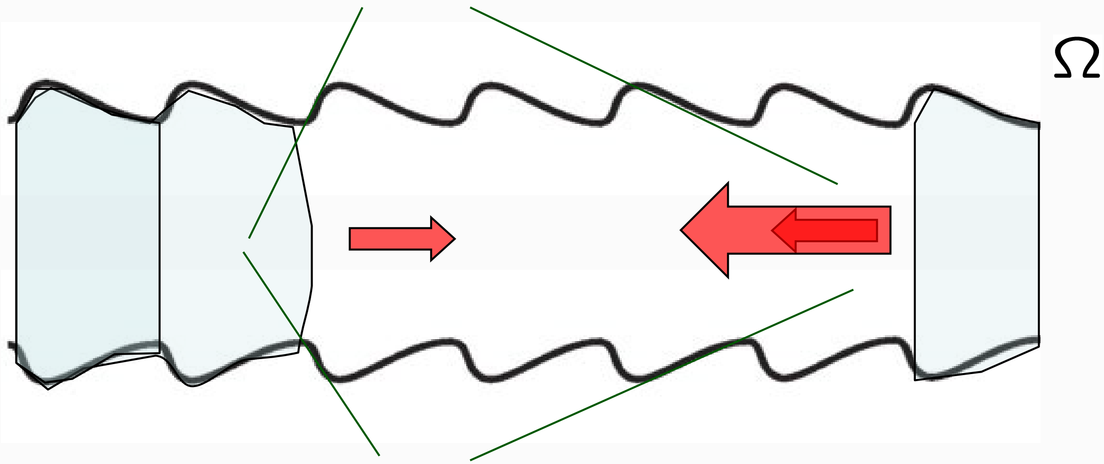
Effect of boundary geometry on front propagation



Q Propagation: Which direction is faster?

Theme 2

Effect of boundary geometry on front propagation

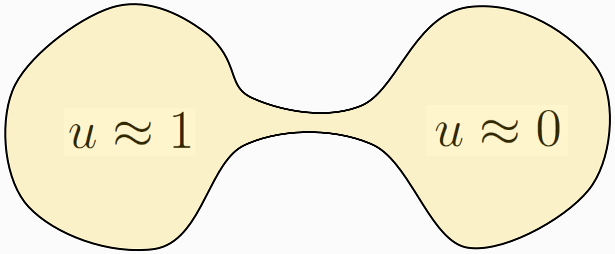


The bigger the opening angles, the slower the speed

Q Propagation: Which direction is faster?

<u>Related problem</u>:

Existence of a non-constant stable stationary solution in a dumbbell-shaped domain



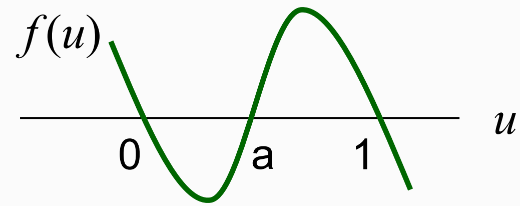
M (1979)

M – Mimura (1983) competition sys

RD equation

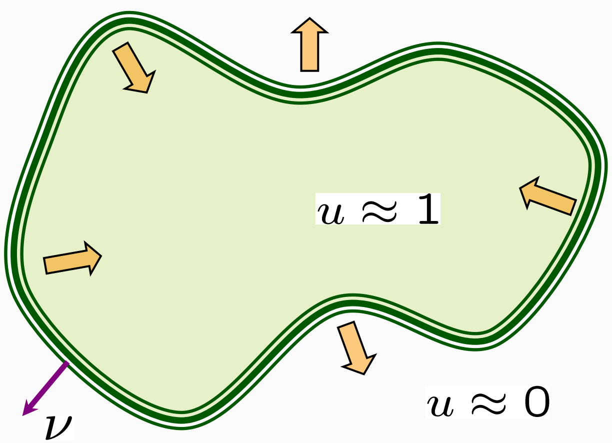
$$\frac{\partial u}{\partial t} = \Delta u + f(u)$$

$f(u)$: bistable



balanced: $\int_0^1 f(s)\,ds = 0$ unbalanced: $\int_0^1 f(s)\,ds > 0$ or $\int_0^1 f(s)\,ds < 0$

Interface motion (N > 1)



outward normal vector

Approximate law of motion

(A) f: balanced $V = (N - 1)H$

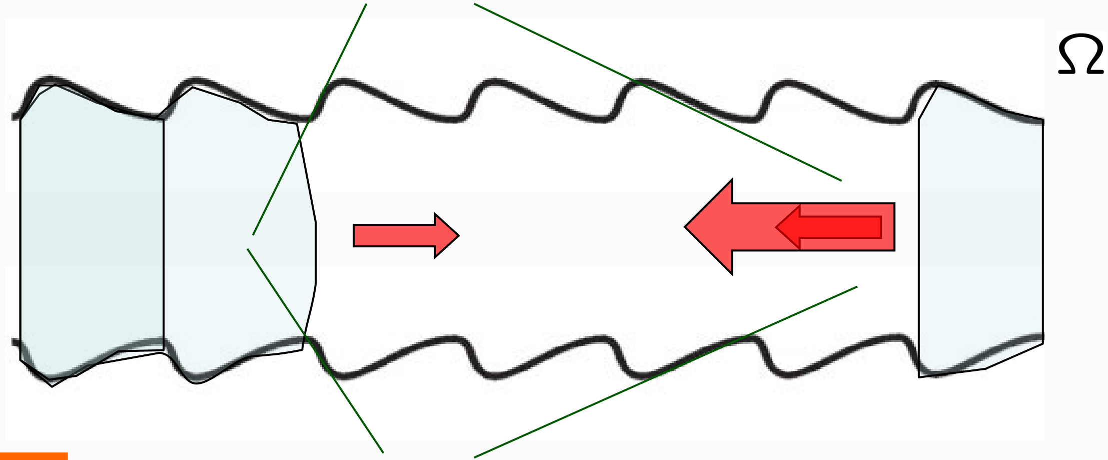
mean curvature flow

(B) f: unbalanced $V = (N - 1)H + A$

V = normal velocity
H = mean curvature
N = space dimension
A : driving force

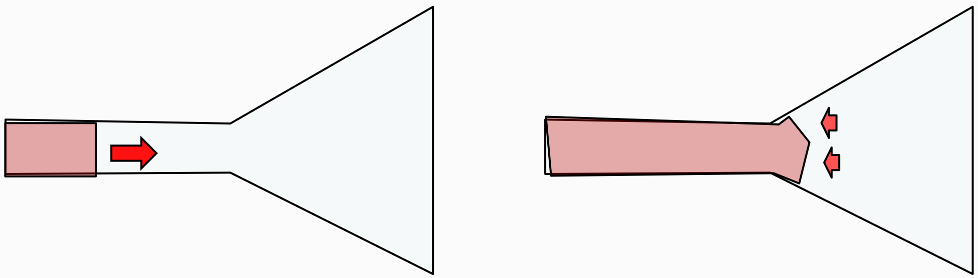
S. Allen & J. Cahn (1979)
K. Kawasaki & T. Ohta (1982)

Effect of boundary geometry on front propagation



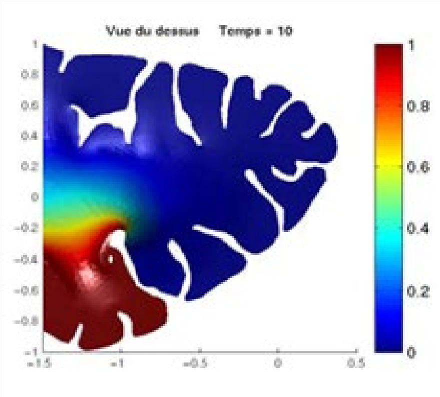
The bigger the opening angles, the slower the speed

Q Propagation: Which direction is faster?

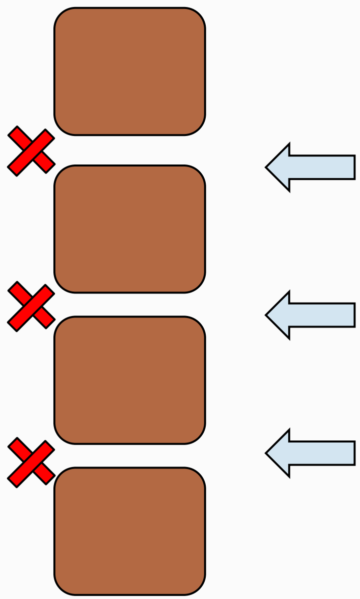


Too rapid widening blocks propagation.

The same fact was used, e.g., in the study of spreading depression (SD) by Dronne *et al* (2004).



Related result. If the holes are too small, then blocking occurs.

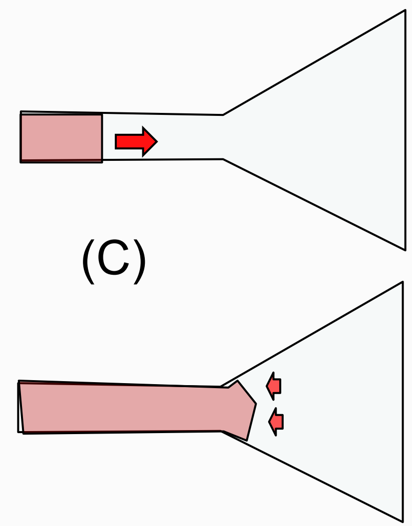


Idea of proof. Either by a variational method or by a super-solution method.

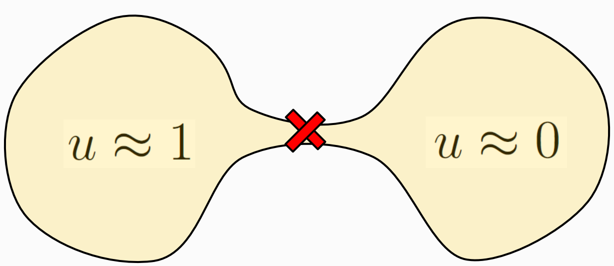
cf. [M. 1979], [M.-Mimura 1983] (A)

[Beres.-Hamel-M. 2009] (B)

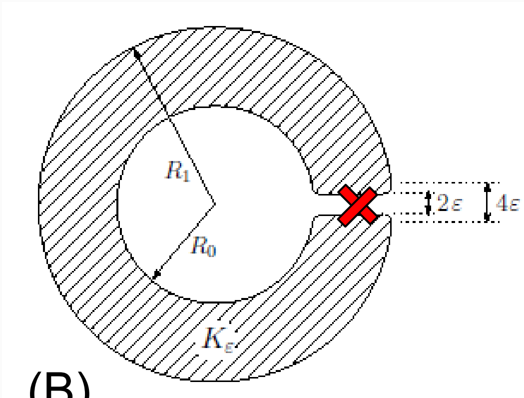
[Berestycki-Bouhours-Chapuisat 2016] (C)



(C)



(A)

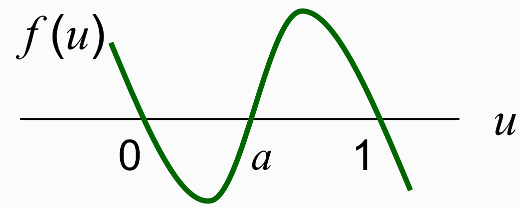


(B)

2. Propagation through a perforated wall

Joint work with Henri Berestycki (EHESS)
François Hamel (Aix-Marseille)

Formulation of the problem



$f(u)$: bistable

$$(E)\quad \begin{cases} u_t = \Delta u + f(u), & x \in \Omega := \mathbb{R}^N \setminus K \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

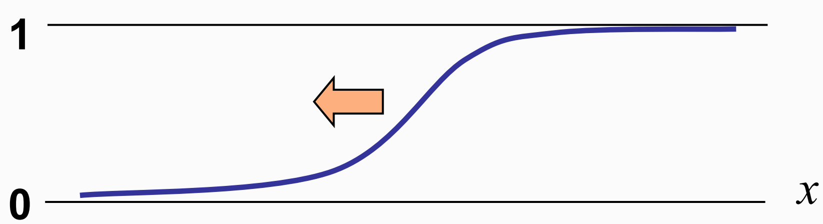
$$f(0) = f(a) = f(1) = 0, \quad f'(0) < 0, \; f'(1) < 0$$

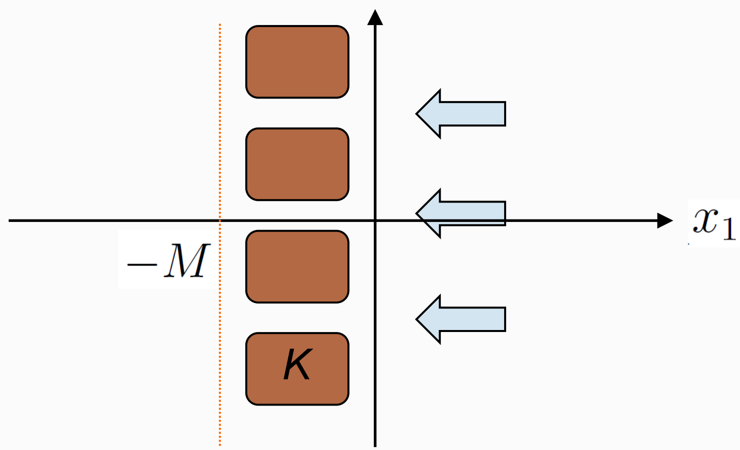
$$\int_0^1 f(s)ds > 0$$

Planar traveling front

$$\phi(x_1 + ct)$$

$$\begin{cases} \phi''(z) - c\phi'(z) + f(\phi(z)) = 0 \quad (z \in \mathbb{R}), \\ 0 < \phi < 1, \quad \phi(-\infty) = 0, \; \phi(+\infty) = 1. \end{cases}$$





Q Under what conditions can the front penetrate through the wall?

Tokyo 2023

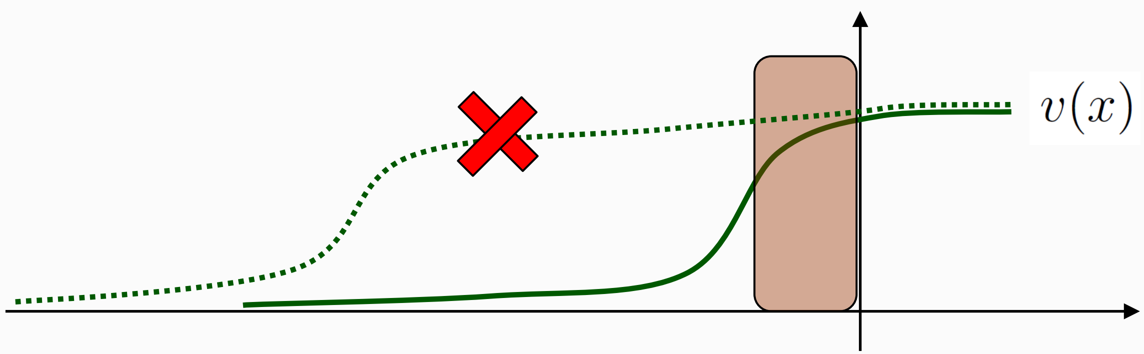
Invasion / blocking dichotomy

Theorem 1 (Dichotomy). *One of the following alternatives holds for the limit profile:*

$$\lim_{x_1\to-\infty} v(x_1,y)=1 \quad \text{(invasion)}, \qquad \lim_{x_1\to-\infty} v(x_1,y)=0 \quad \text{(blocking)}$$

The above convergence is uniform with respect to $y\in\mathbb{R}^{N-1}$ and K so long as $K\subset\{x\in\mathbb{R}^N \mid -M\le x_1\le 0\}$.

In particular, there is no blocking profile that converges to 0 too slowly.



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The above convergence is uniform with respect to $y \in \mathbb{R}^{N-1}$ and K so long as $K \subset \{x \in \mathbb{R}^N \mid -M \le x_1 \le 0\}$.

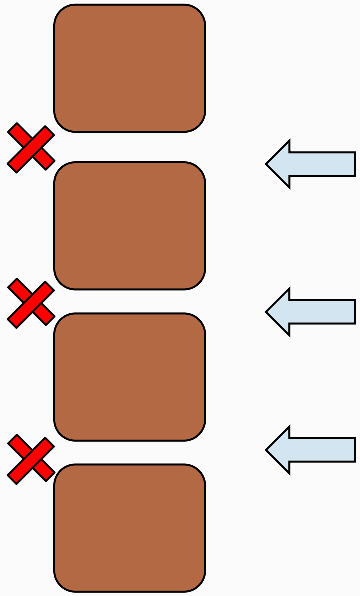
Corollary 2. *Let $K_1, K_2, K_3, \ldots$ be a sequence of walls satisfying*

$$K_i \subset \{x \in \mathbb{R}^N \mid -M \le x_1 \le 0\} \quad (i = 1, 2, 3, \ldots)$$

and converging to a wall K in a certain appropriate sense. If blocking occurs for every K_i $(i = 1, 2, 3, \ldots)$ then the same holds for K.

Dichotomy theorem ⟸ De Giorgi type lemma

Theorem 3. If the holes are too small, then blocking occurs.



Idea of proof. Either by a variational method or by a super-solution method.

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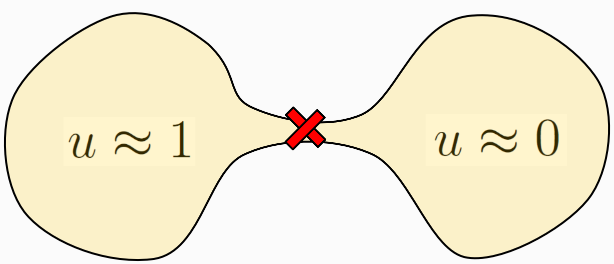
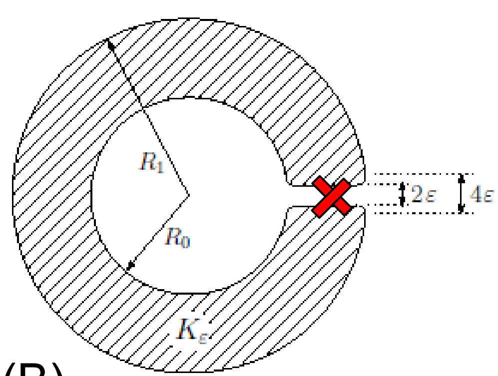
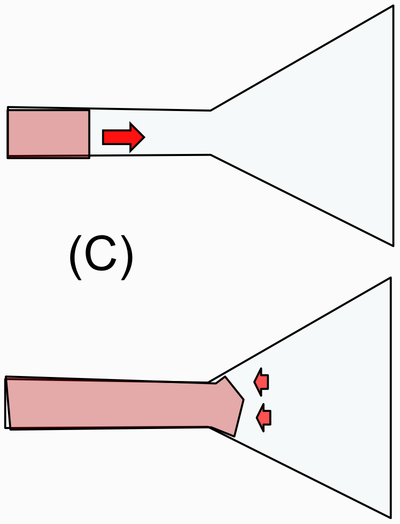


(A)



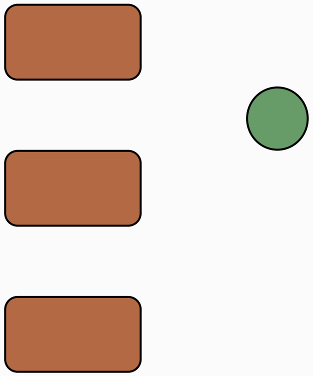
(B)



(C)

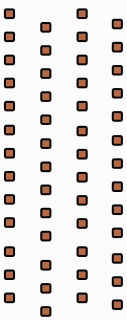
Three types of walls

1. Wall with large holes



A ball of radius $\boldsymbol{R}_0$ can pass through one of the holes in the wall, where $\boldsymbol{R}_0$ is a certain positive number to be specified later.

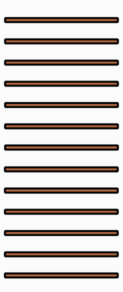
2. Small capacity wall



K is close to a set of capacity 0 in a certain sense.

(K_ε is in the ε neighborhood of a set K_0 of capacity 0.)

3. Skeleton wall



K consists of thin panels parallel to the x_1 axis.
More precisely, K_0 is a locally finite union of hypersurfaces parallel to the x_1 axis and let K_ε converge to K_0 in a certain sense.

3. Propagation in a saw-toothed cylinder

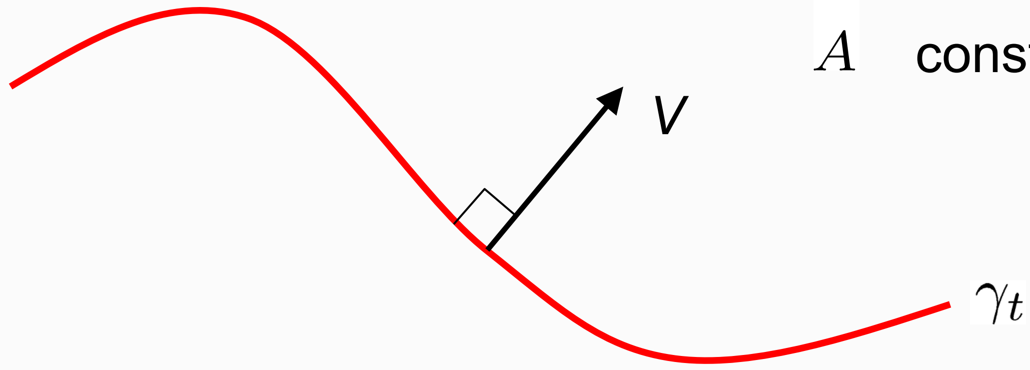
Curvature-dependent motion of interfaces

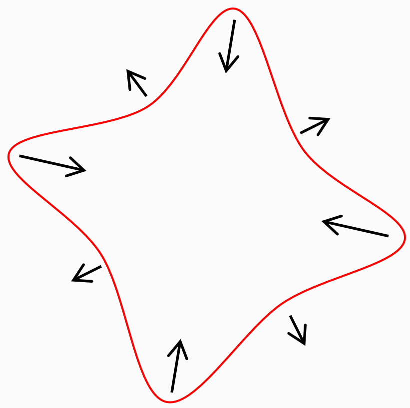
Joint work with Ryunosuke Mori (Meiji University)

Curvature-dependent motion of a plane curve

$$V = \kappa + A$$

V normal velocity
κ curvature
A constant > 0





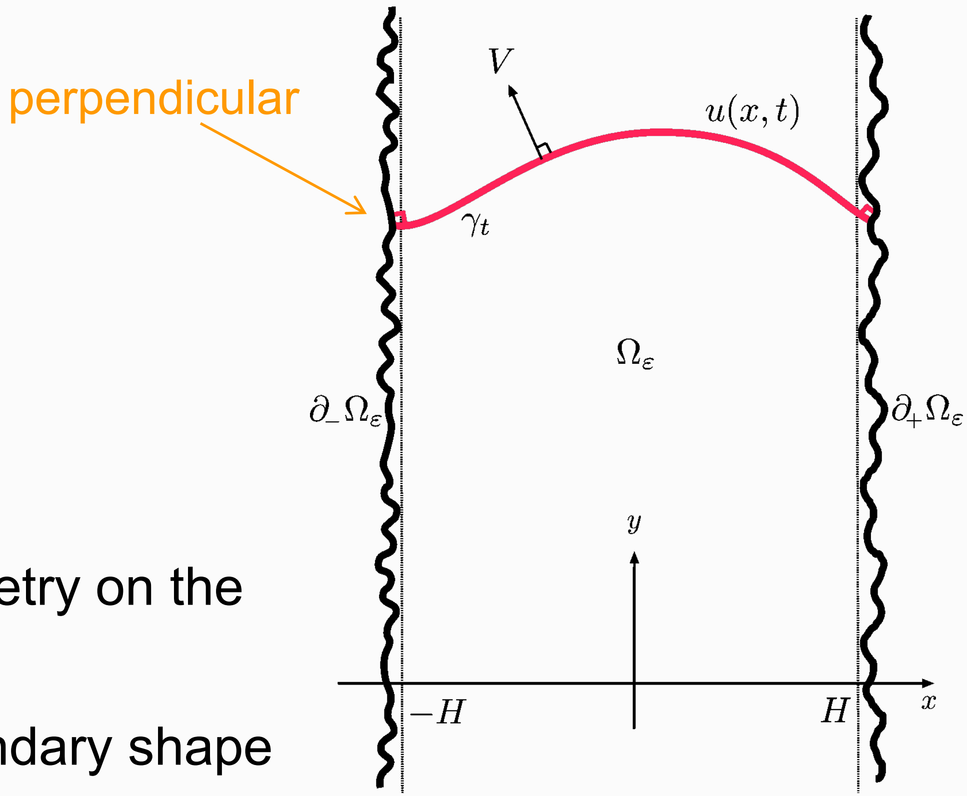
- Evolution of a phase boundary.
- Singular limit of the Allen-Cahn equation

$$\partial_t u = \Delta u + f(u)$$

Motion in an infinite strip with undulating boundaries

$$V = \kappa + A$$

V	normal velocity
κ	curvature
A	constant > 0



Motivation

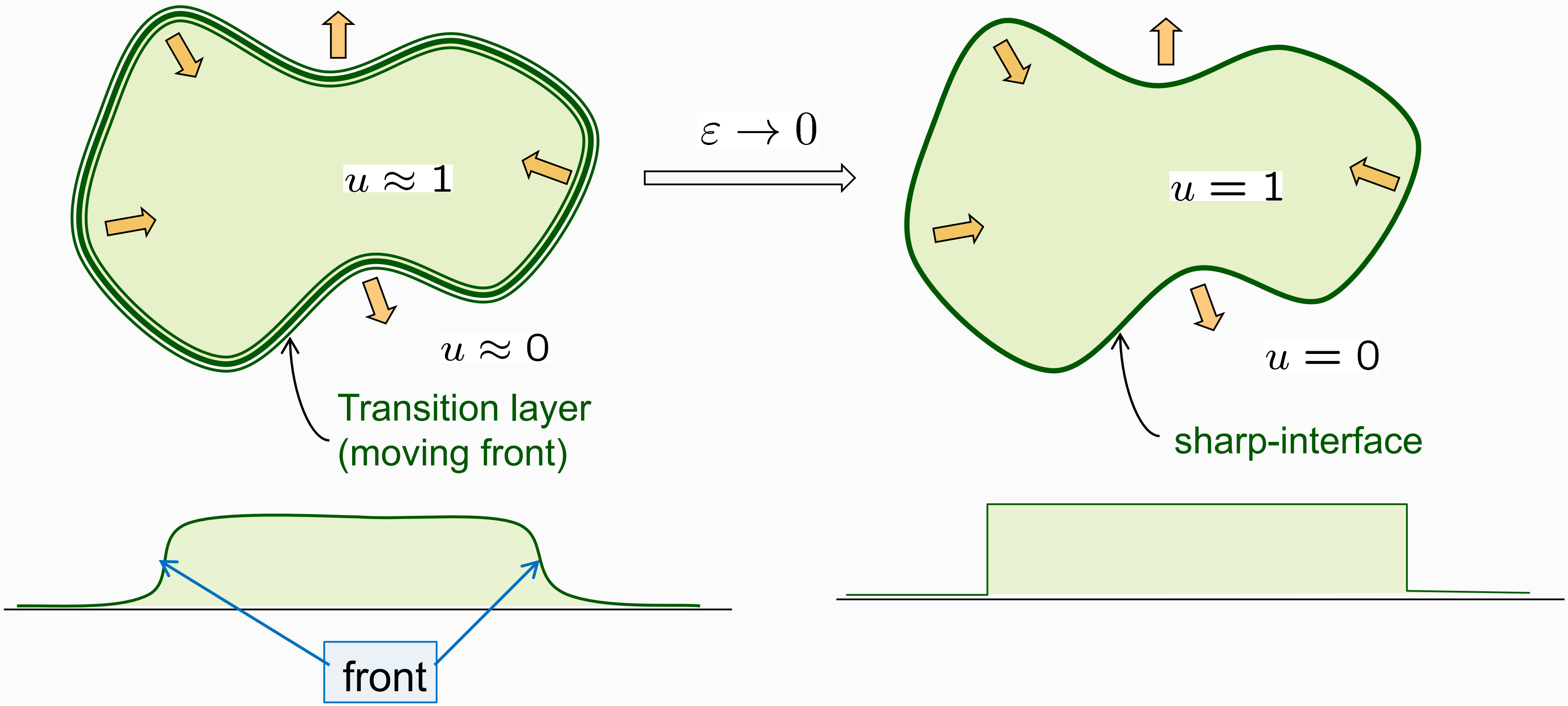
1. To study the effect of geometry on the speed of propagation.
2. Find conditions on the boundary shape for propagation and blocking.

New discovery: obstacle-induced propagation

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{\varepsilon^2}\big(f(u) - \varepsilon g(u)\big)$$

Singular limit ⟹ mean curvature flow

sharp-interface limit



Typical expample

$$\frac{\partial u}{\partial t} = \Delta u + \frac{C}{\varepsilon^2} u(1-u)\left(u - \frac{1}{2} - \varepsilon a\right) \Rightarrow V = \kappa + A$$

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2. Previous results

Motion without singularity
(the case of mildly undulating boundaries)

[1] HM, K.-I. Nakamura, B. Lou
(Networks & Heterogeneous Media 2006)

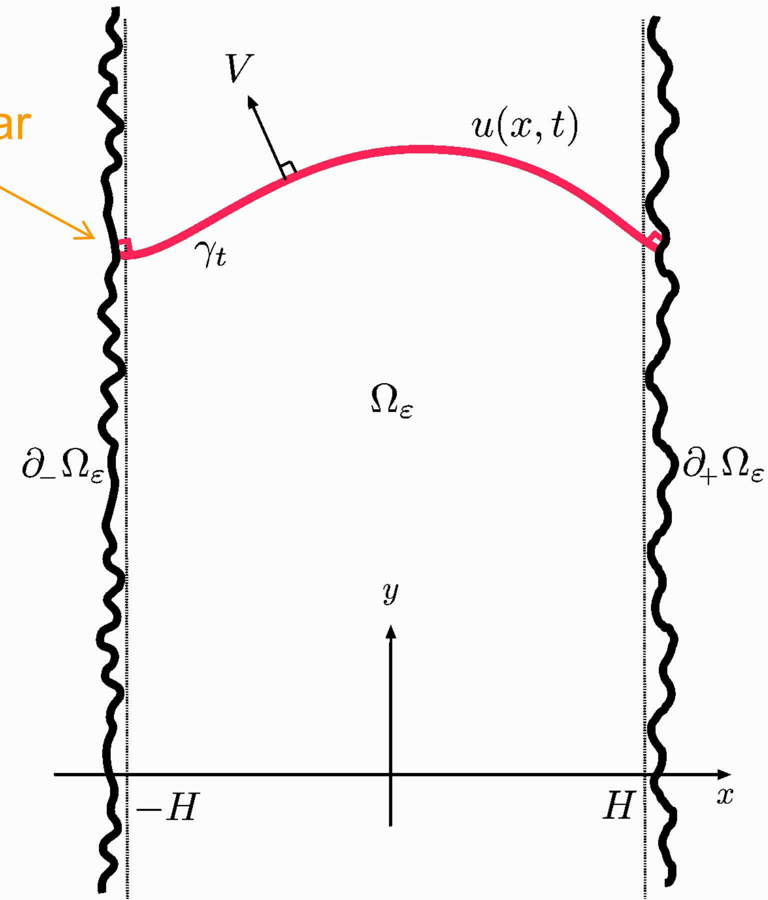
[2] B. Lou, HM, K.-I. Nakamura
(JDE 2013)

$$V = \kappa + A$$

V	normal velocity
κ	curvature
A	constant

$$\Omega_\varepsilon = \{-H - g_-^\varepsilon(y) < x < H + g_+^\varepsilon(y),\ y \in \mathbf{R}\}$$

perpendicular



$$g_\pm^\varepsilon(y) = \varepsilon g_\pm(y/\varepsilon)$$

$g_\pm(y)$ recurrent functions

If the curve is a graph: $y = u(x,t)$
then the equation reduces to:

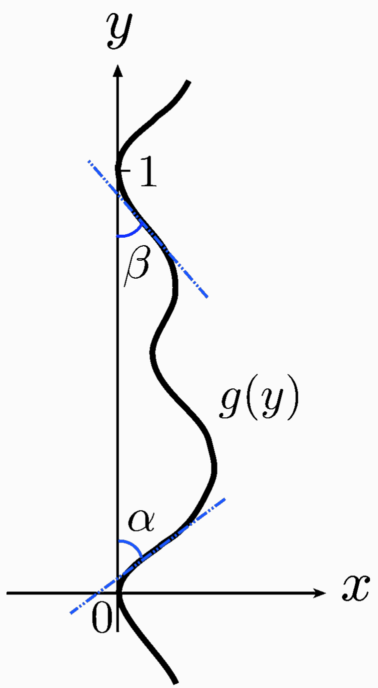
$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}$$

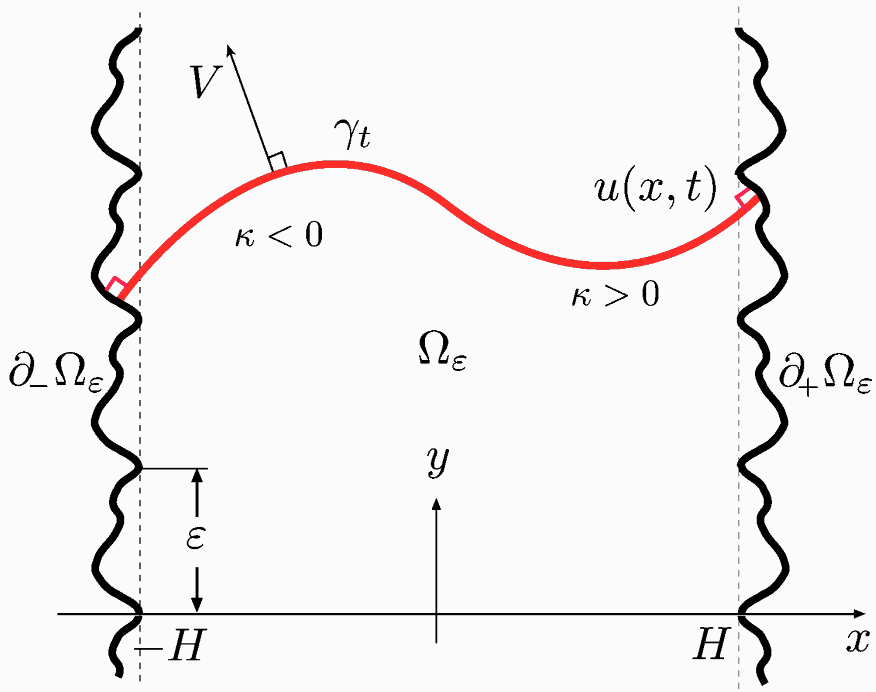
$$u_x(\zeta_\pm(t),t) = \mp g_\varepsilon'(u(\zeta_\pm(t),t)),\ \ (\zeta_\pm(t),u(\zeta_\pm(t),t)) \in \partial_\pm\Omega_\varepsilon$$

Notation

$$\alpha_\pm = \sup_y \, g'_\pm(y) \quad \text{maximal opening angles}$$

$$\beta_\pm = -\inf_y \, g'_\pm(y) \quad \text{maximal closing angles}$$





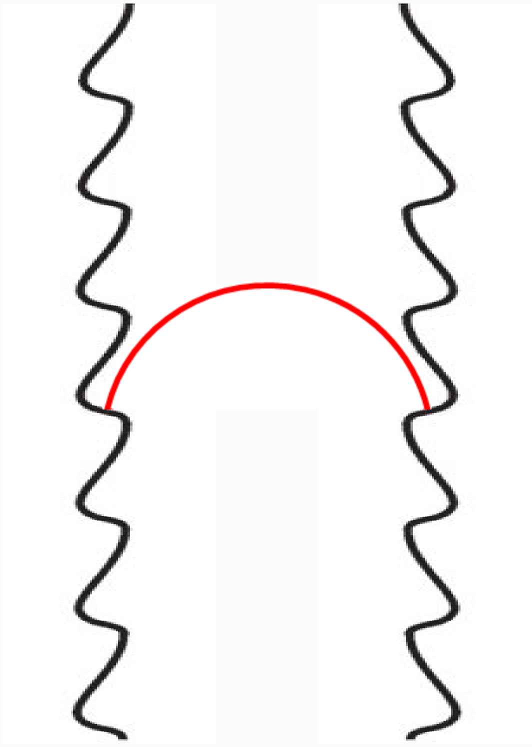
Stationary solution

$$V = \kappa + A$$

γ : stationary

$\Longleftrightarrow$

γ is a circular arc of curvature $-A$ whose ends meet the boundary with the angle $\pi/2$.



Definition

Blocking: The curve remains bounded as $t \to \infty$.

Propagation: The curve (or at least its portion goes to infinity in y direction as $t \to \infty$.

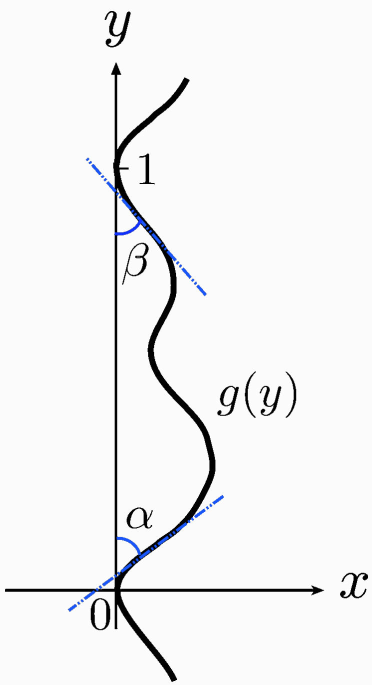
Slope condition in the previous works

[1] M, Nakamura & Lou (2006)

[2] Lou, M & Nakamura (JDE 2013)

$$0 \leq \alpha_\pm,\ \beta_\pm < \frac{\pi}{4}$$

Boundary slope condition



Assumption on the initial curve $|u_x(x,0)| < 1$

⇩

A unique classical solution of the following problem exists globally in time and satisfies $|u_x(x,t)| < 1 \ ({}^\forall t \geq 0)$

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}$$ +B.C.

Motion without singularity

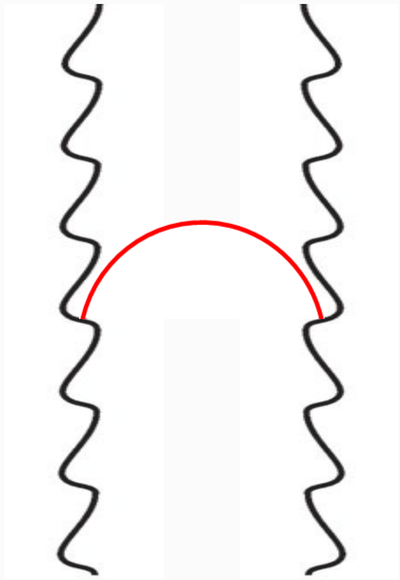
The results of MNL (2006)

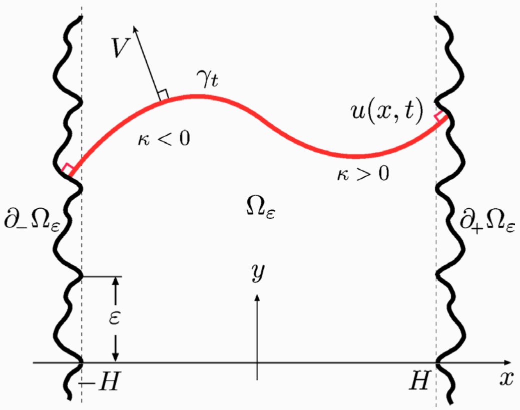
$g(y)$ periodic

1. Blocking occurs if and only if there is a stationary solution, or, equivalently, if and only if $2AH < \sin\alpha_- + \sin\alpha_+$
   In this case, any solution converges to a stationary solution as $t \to \infty$.

2. If no stationary solution exists, then there exists a traveling wave solution, which is unique up to time shift. In this case, any solution converges to a traveling wave as $t \to \infty$.

3. Results on the homogenization limit.





Idea of proof: Strong maximum principle and the analysis of the omega limit set.

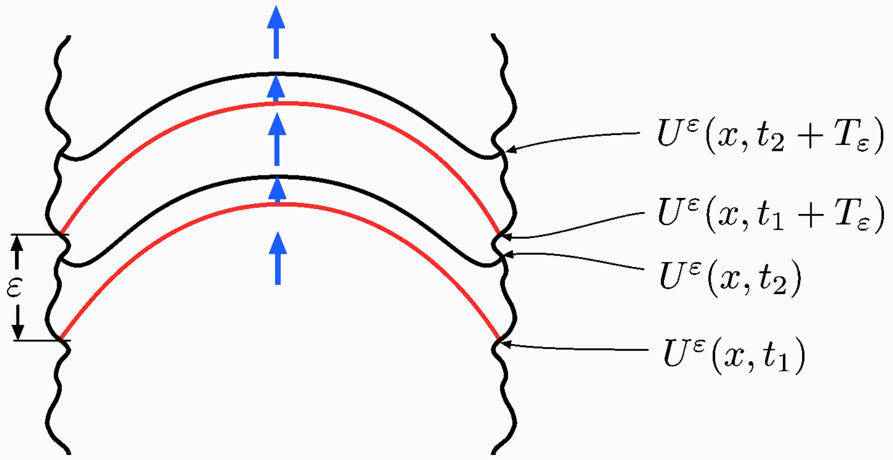
What are traveling waves?

The periodic case: $g_\pm(y+L) \equiv g_\pm(y)$, i.e. $g_\pm^\varepsilon(y+\varepsilon L) \equiv g_\pm^\varepsilon(y)$

Definition (periodic TW):

$$U^\varepsilon(x, t+T_\varepsilon) = U_\varepsilon(x,t) + \varepsilon L$$

$c_\varepsilon := \varepsilon L \,/\, T_\varepsilon$ average speed



Remark: In the non-periodic case, traveling waves can be defined by using the notion of hull of a function.

Homogenization

$$g_{\pm}^{\varepsilon}(y) = \varepsilon g_{\pm}(y/\varepsilon)$$

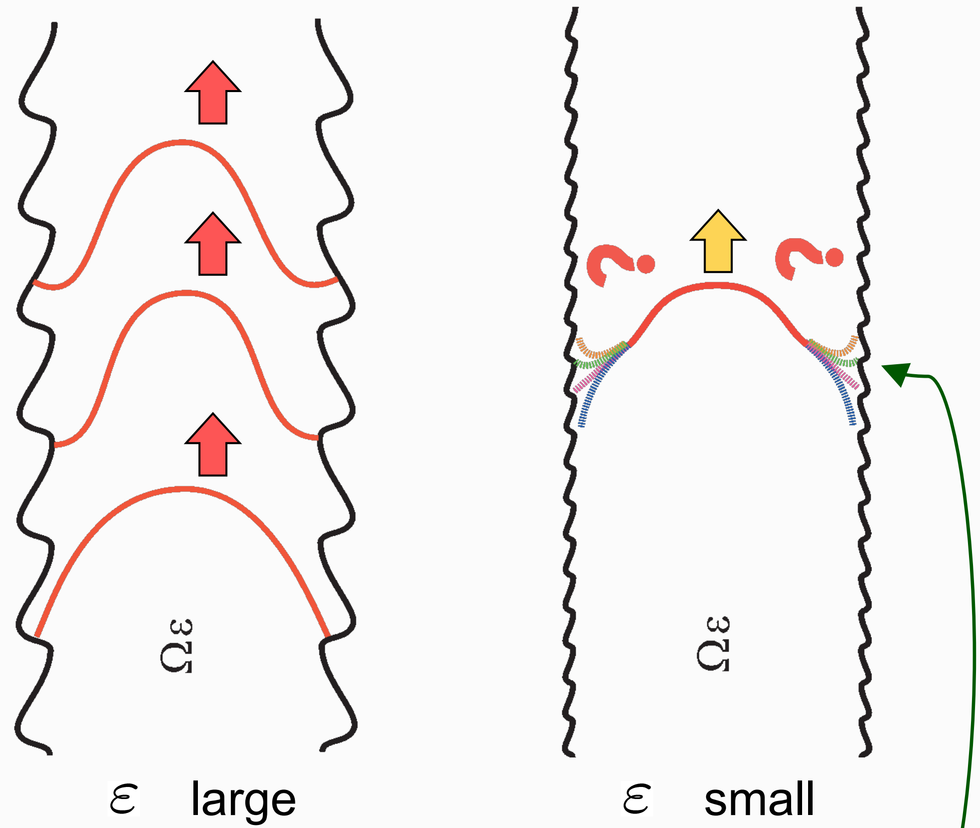
boundary shape

Homogenization limit

$$\gamma^{\varepsilon}(t) \to \varphi(x) + c_0 t \quad (\varepsilon \to 0)$$

c_0: limit speed

φ: limit profile



What determines the limit speed ?

The limit contact angle plays the key role.

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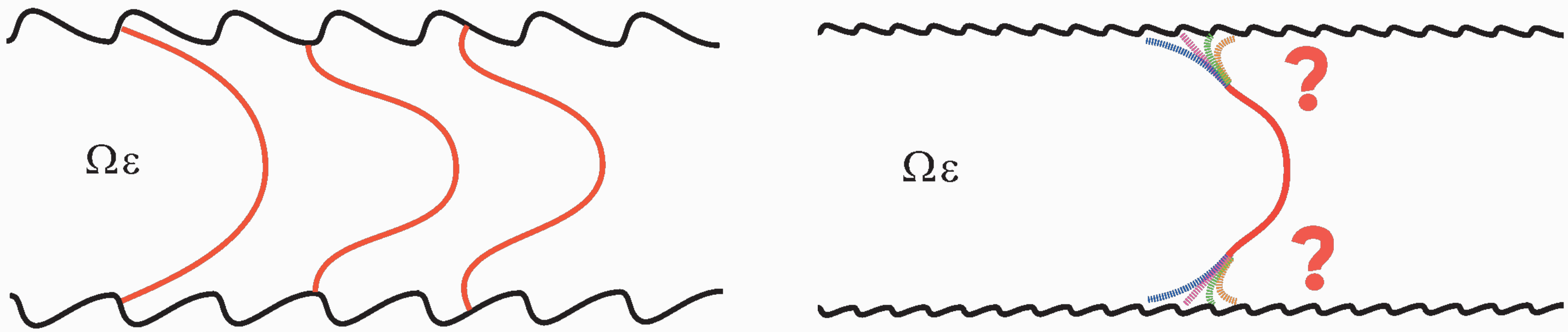
$$g_i^\varepsilon(y) = \varepsilon g_i(y/\varepsilon) \to 0$$



Difficulty: The two ends of the curve flips back and forth very rapidly, which makes it difficult to derive a precise asymptotic expansion even formally.

Strategy

Estimate the gradient at $\sqrt{\varepsilon}$ away from boundary.

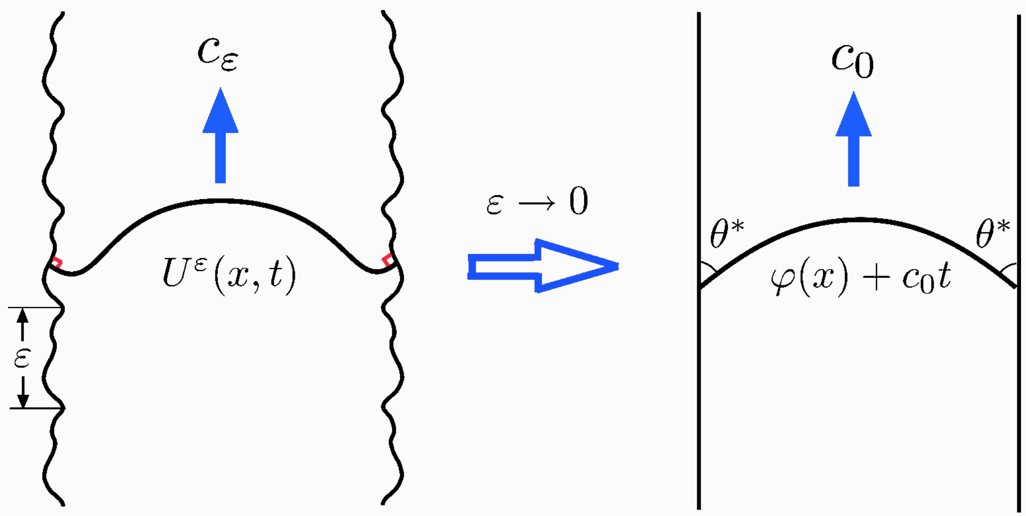
Homogenization limit

Theorem [1] (homogenization).

Assume $AH > \sin\alpha$ and let $U^\varepsilon(x,t)$ be the periodic TW that is normalized to satisfy $U^\varepsilon(0,0)=0$. Then

(i) $U^\varepsilon(x,t)$ converges to a function of the form $\varphi(x)+c_0 t$ as $\varepsilon\to 0$ whose contact angle is $\theta^* = \pi/2-\alpha$.

(ii) The limit speed c_0 is determined by $H = \int_0^\alpha \frac{\cos\eta}{A - c_0\cos\eta}\,d\eta.$

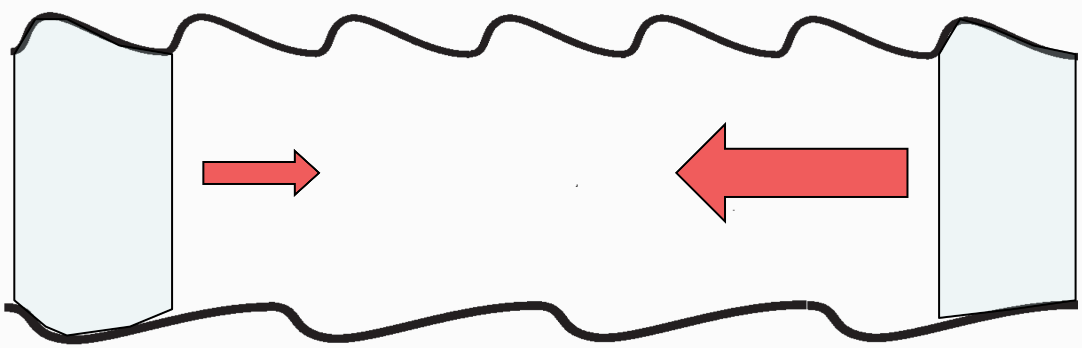


$$H = \int_0^{\alpha} \frac{\cos\eta}{A - c_0 \cos\eta} d\eta.$$

Corollary. The limit speed c_0 satisfies

$$\frac{\partial c_0}{\partial \alpha} < 0, \qquad \frac{\partial c_0}{\partial A} > 0, \qquad \frac{\partial c_0}{\partial H} > 0.$$

The larger the
opening angle α,
the slower
the speed c_0 .



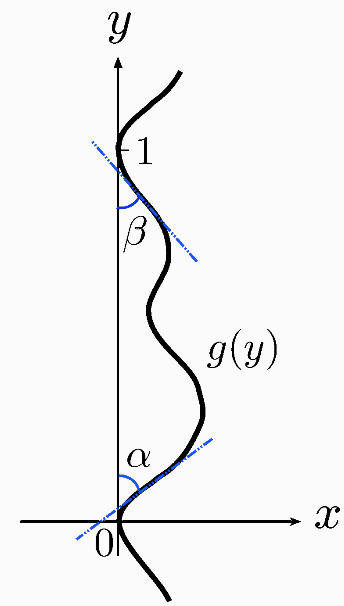
3. Main results

- Traveling waves with singularities
- Obstacle-induced propagation

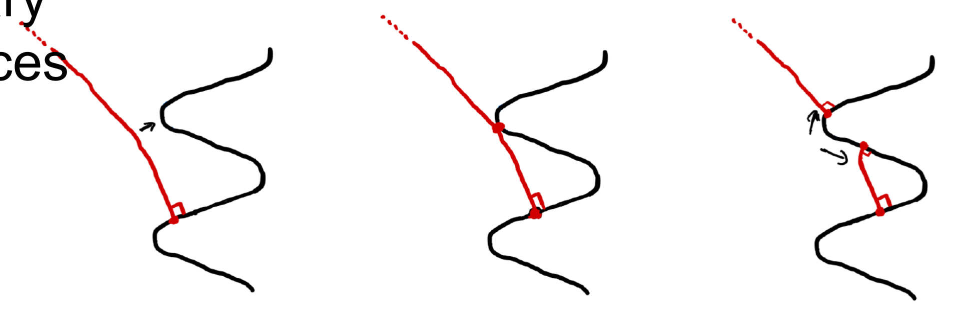
Removal of the slope condition

$$0 \le \alpha_\pm, \beta_\pm < \frac{\pi}{4}$$

Boundary slope condition



The curve may touch the boundary besides the endpoints, which forces the curve to split, thus creating singularities.



The solution can no longer be treated in the classical framework, so we consider the problem in the framework of viscosity solutions.

Viscosity solution

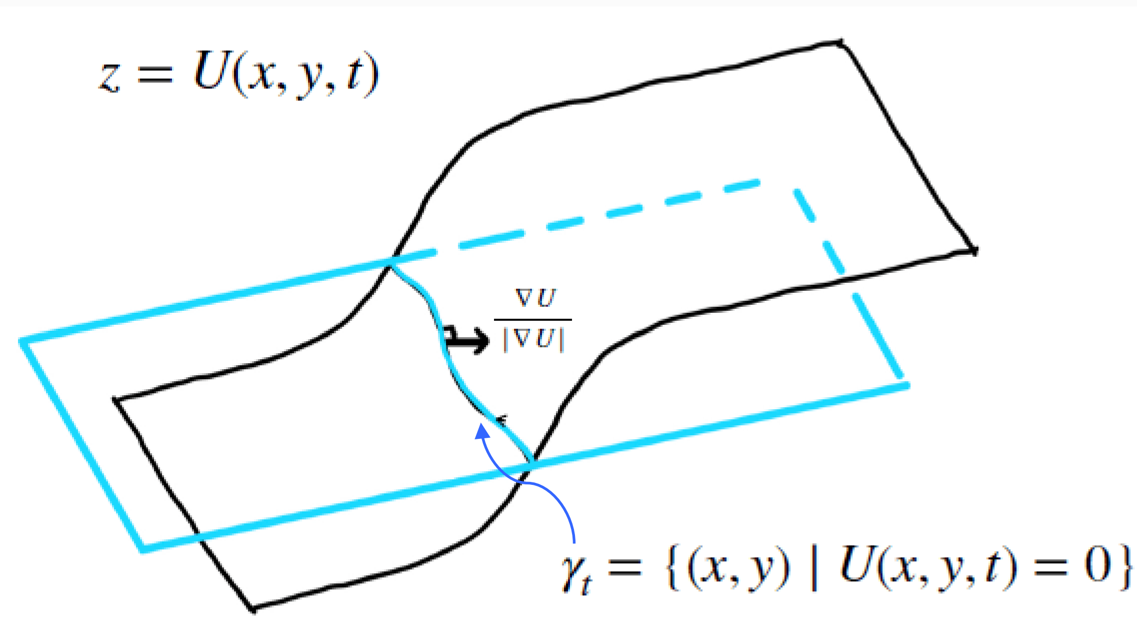
Classical setting

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}$$

$$u_x(\zeta_\pm(t),t) = \mp g_\varepsilon'(u(\zeta_\pm(t),t)), \ (\zeta_\pm(t), u(\zeta_\pm(t),t)) \in \partial_\pm \Omega_\varepsilon$$

Level set approach

Regard the curve γ_t as a level set of an auxiliary function $U(x,t)$.



$$\frac{\nabla U}{|\nabla U|} = \text{normal vector},$$

$$V = -\frac{\partial_t U}{|\nabla U|},$$

$$\kappa = -\mathrm{div}\left(\frac{\nabla U}{|\nabla U|}\right).$$

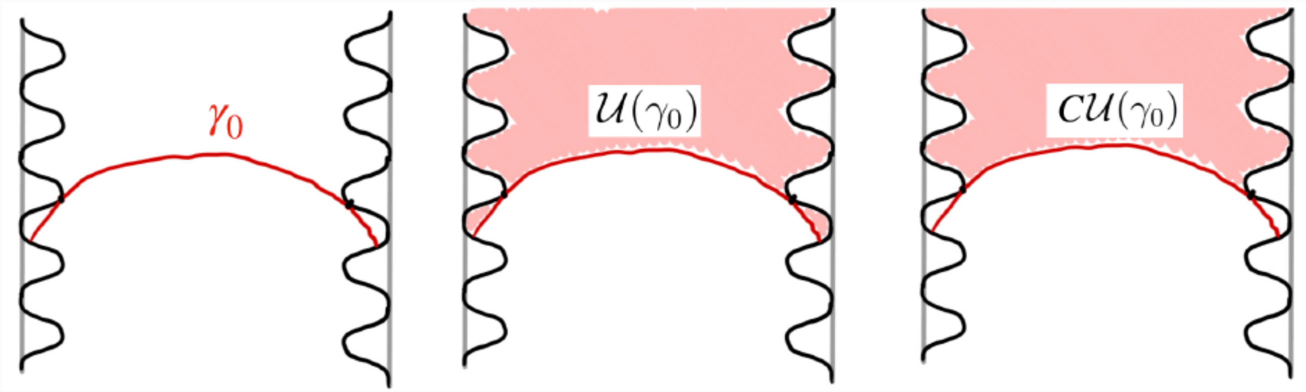
The function U satisfies the following equation formally.

$$\begin{cases} \partial_t U = |\nabla U| \operatorname{div}\left(\dfrac{\nabla U}{|\nabla U|}\right) - A|\nabla U| & \text{in} \quad \Omega, \\ \partial_\nu U = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

We consider viscosity solutions of the above equation and regard it as a solution of the original problem in a generalized sense.

Note: we focus on the central component of the viscosity solution.

$\mathcal{U}(\gamma_0)$: viscosity solution $\mathcal{CU}(\gamma_0)$: central component

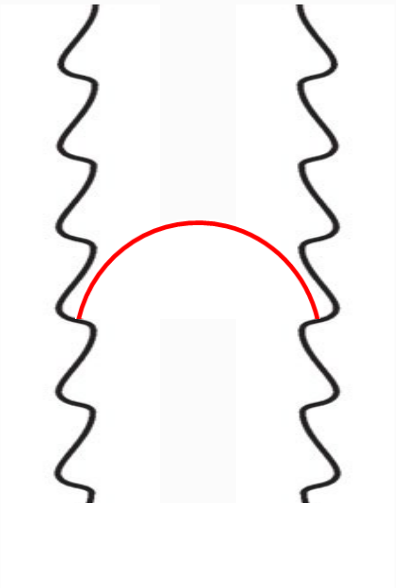


Blocking and propagation

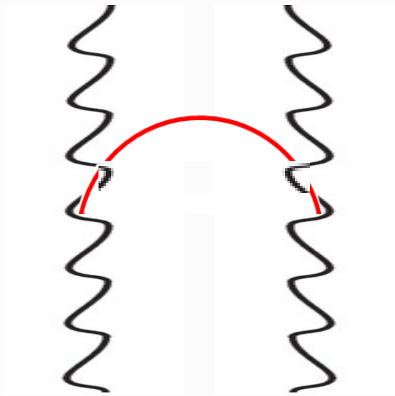
1. Blocking occurs if and only if there is a stationary solution.

2. If no stationary solution exists, then there exists a traveling wave solution, which is unique up to time shift. In this case, any solution without fattening converges to a traveling wave as $t \to \infty$.

3. Results on the homogenization limit.



Corollary. If the bumps are sufficiently dense, then propagation occurs.



No stationary solution

Obstacle-induced propagation

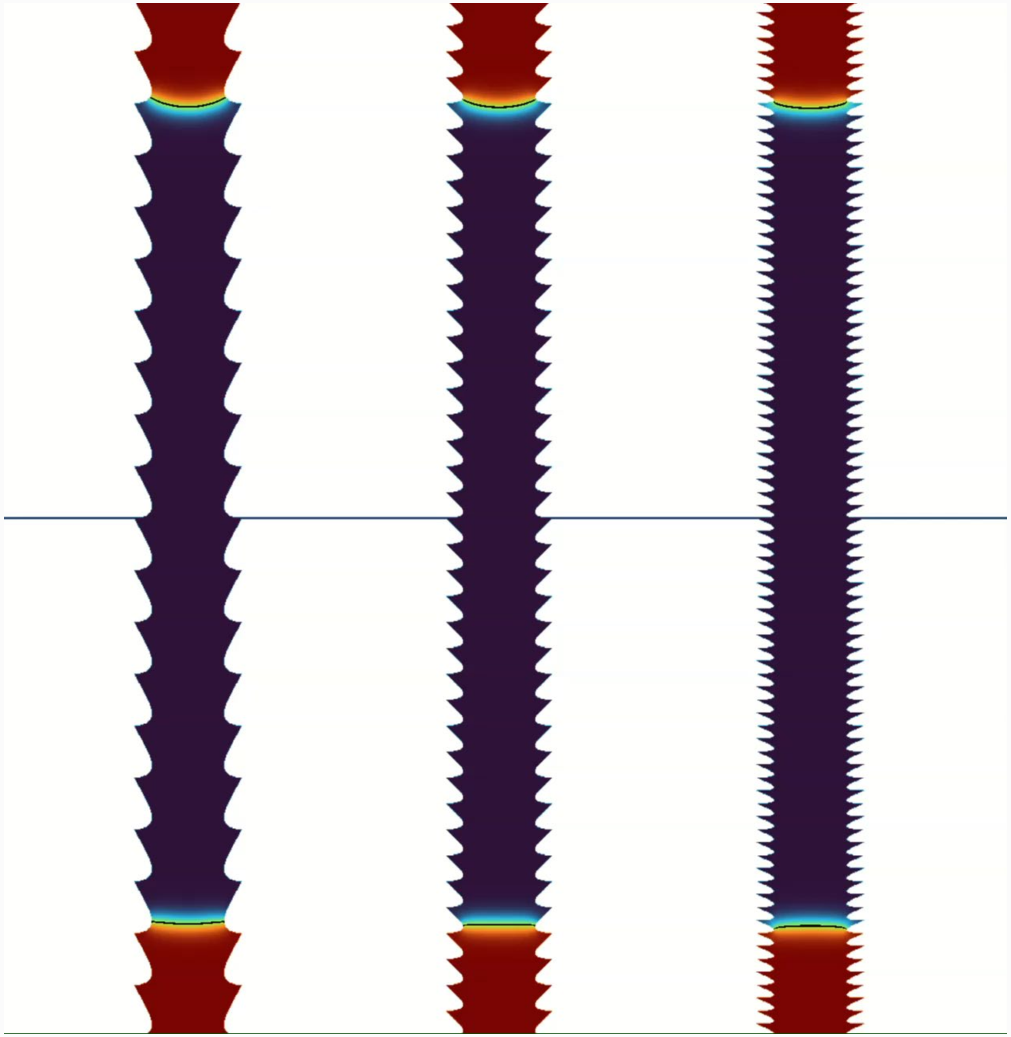
(This idea was proposed by H. Ninomiya for RD.)

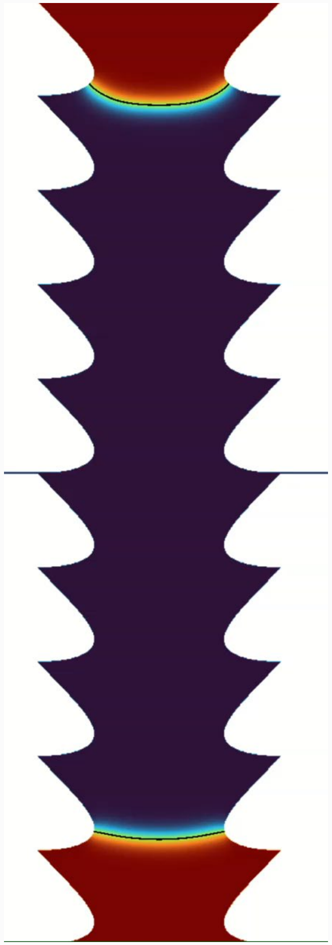
4. Numerical simulation

Propagation with singularities

Which direction is faster?

Simulation by Steffen Plunder (ASHBi, Kyoto University)





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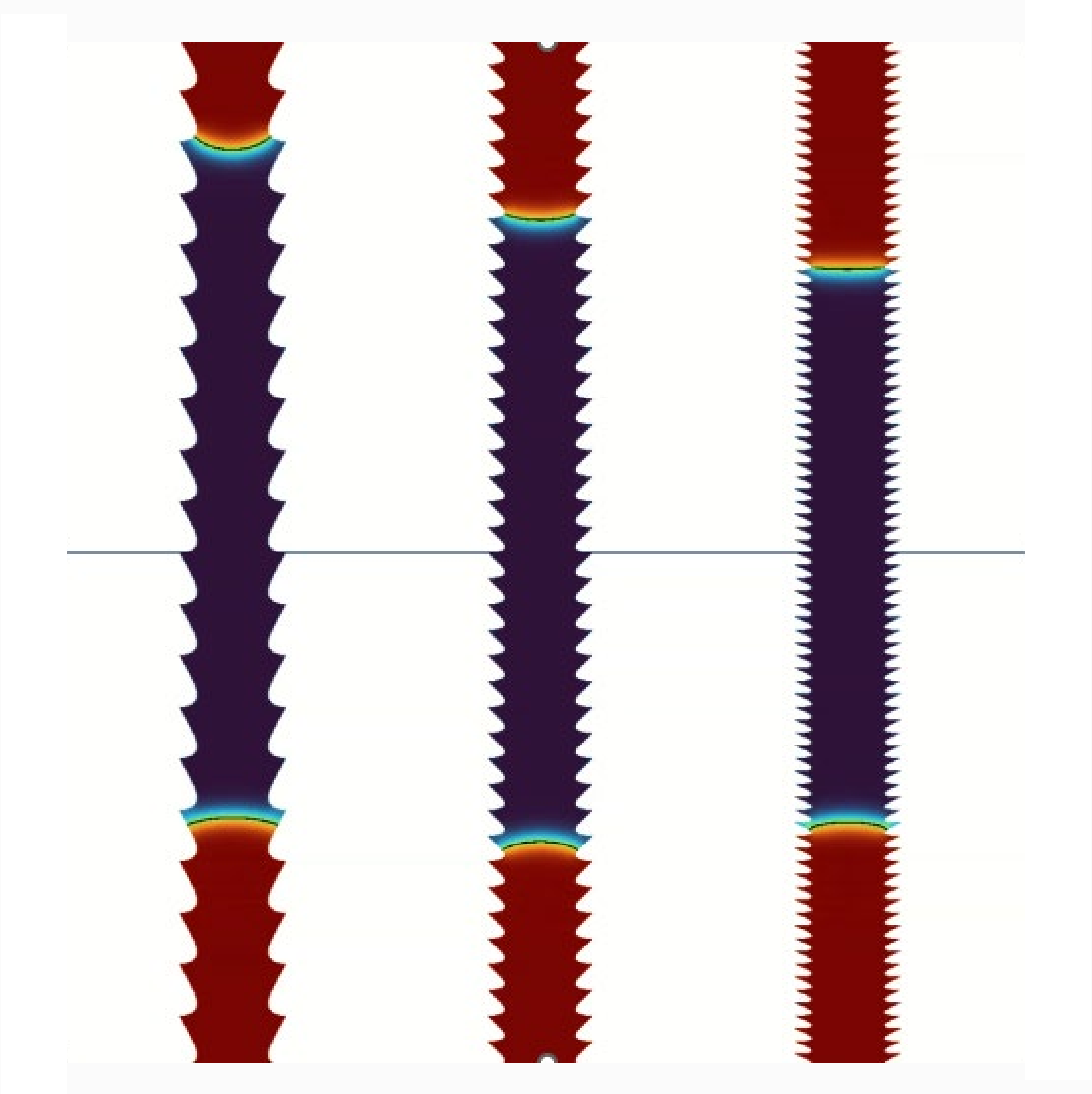
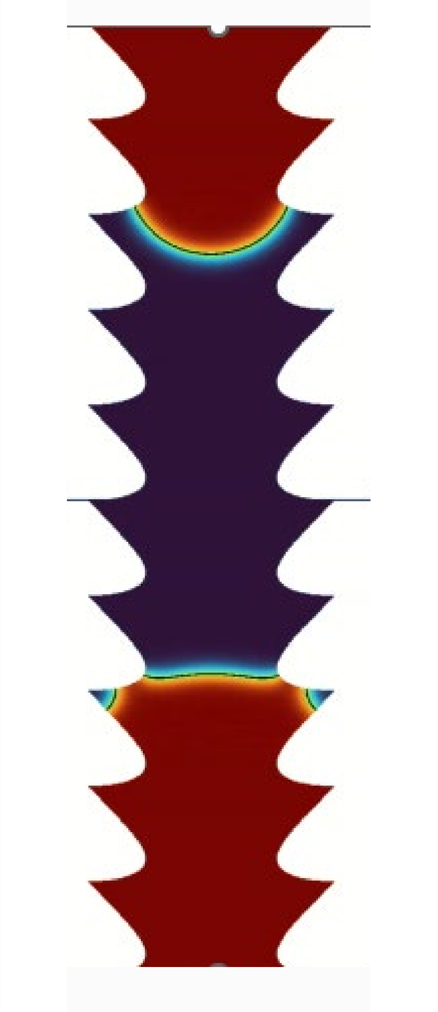
Which direction is faster?

Simulation by Steffen Plunder
(ASHBi, Kyoto University)





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Which direction is faster?

Simulation by Steffen Plunder
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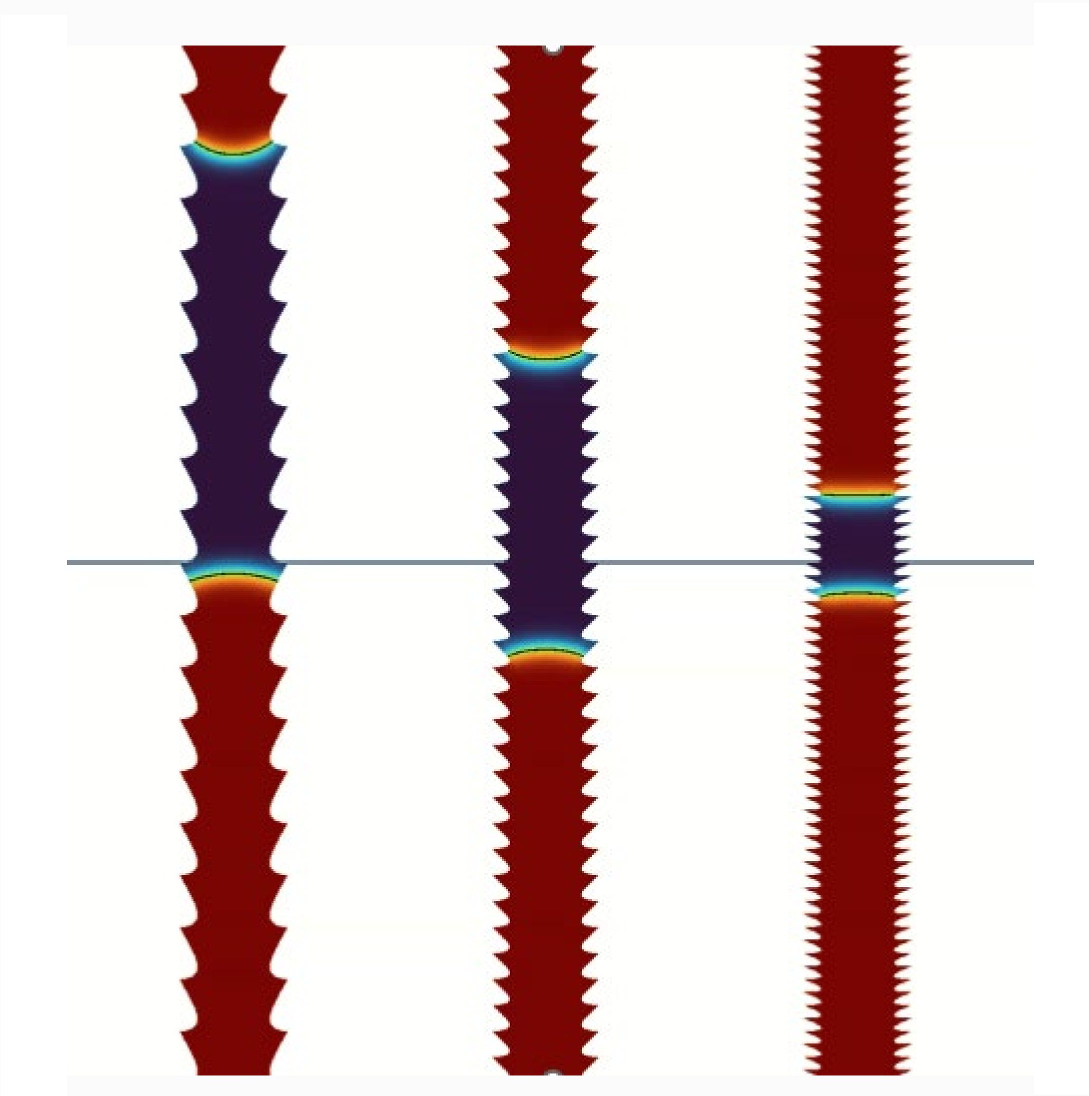

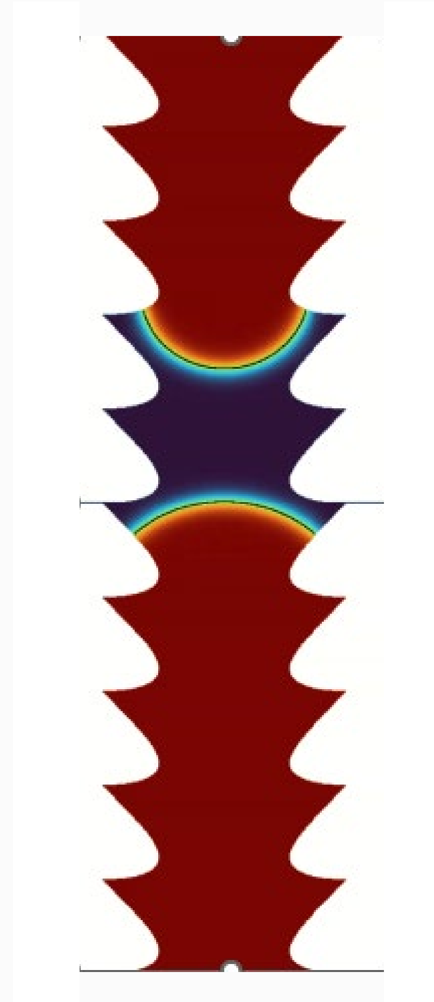

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Obstacle-aided propagation

Simulation by Steffen Plunder
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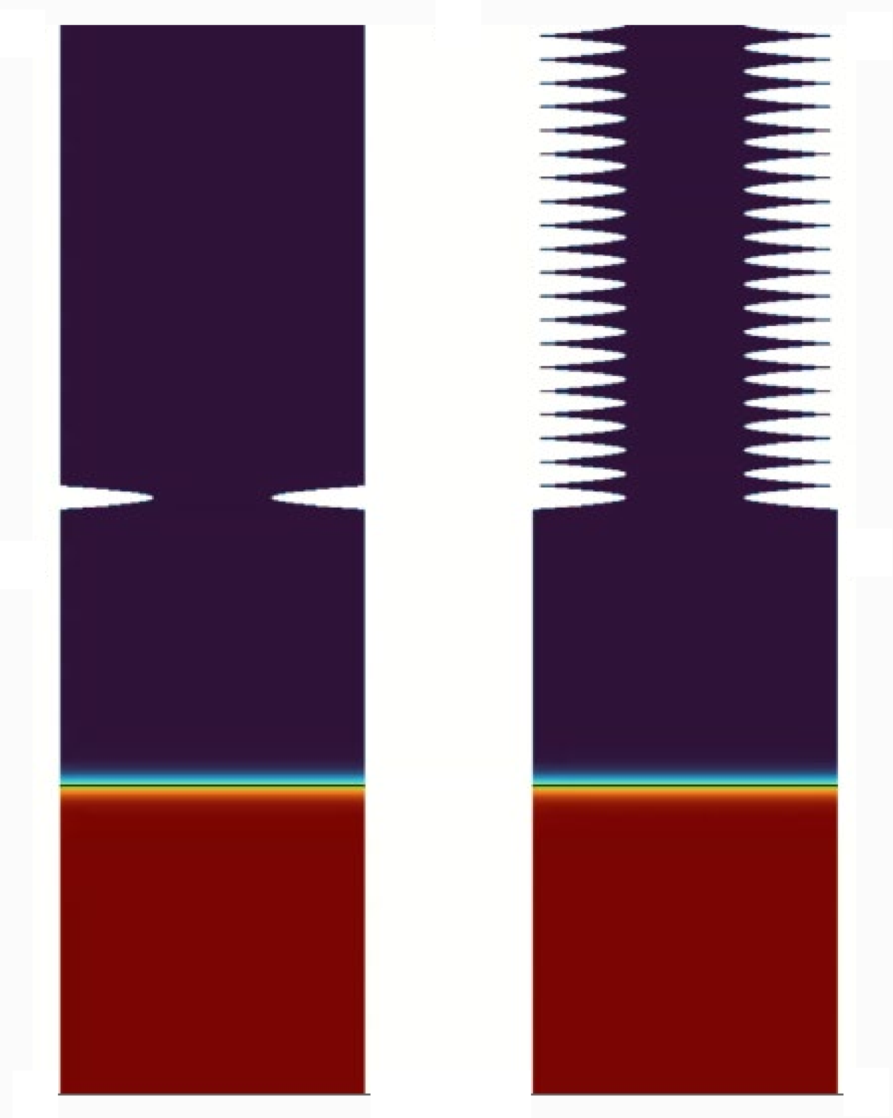

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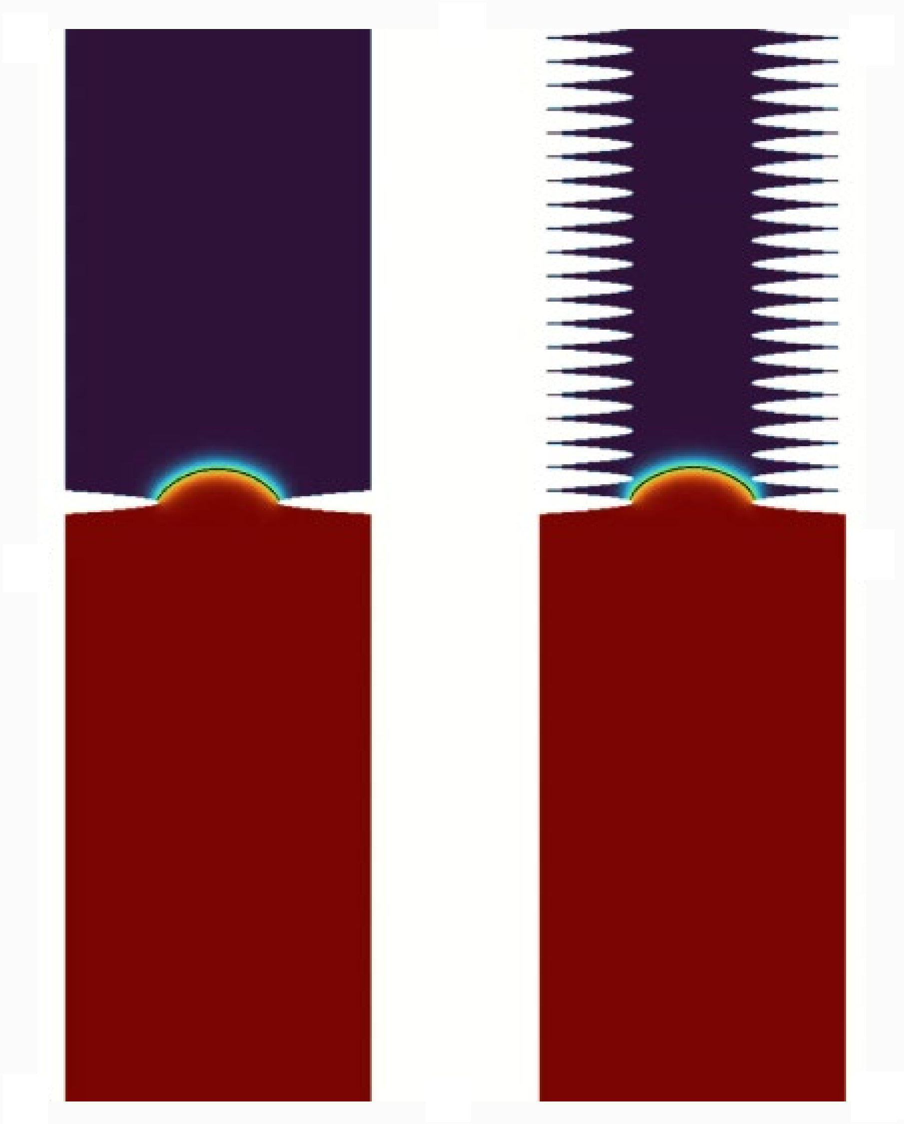
Obstacle-aided propagation

Simulation by Steffen Plunder
(ASHBi, Kyoto University)



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Obstacle-aided propagation

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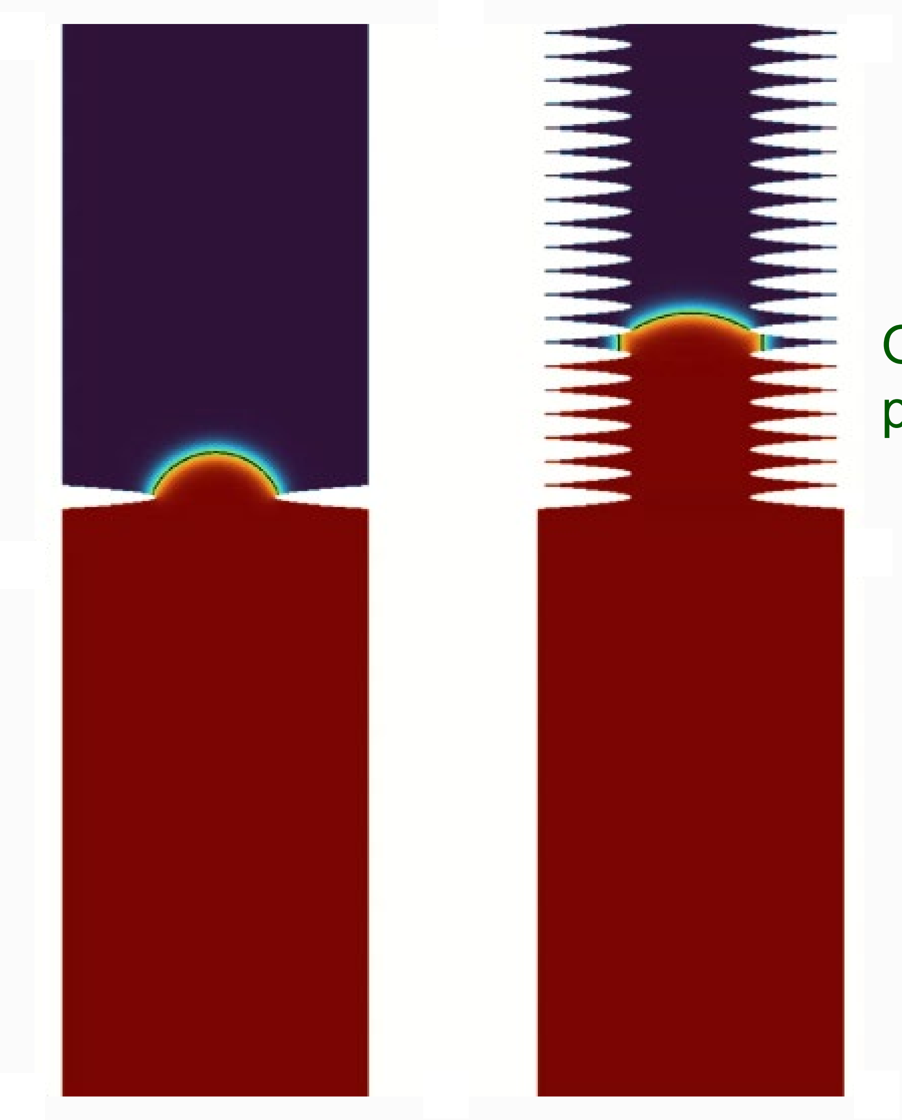

Obstacle-induced propagation

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Thank you for your attention!

ICMMA 2023

Poster Presentation

* Excellent Poster Presentation Award

No.	Poster title	Author(s) Name
*1	"Stationary and oscillatory Turing patterns for reaction-diffusion systems of topological signals coupled by the Dirac operator"	Riccardo Muolo (Tokyo Institute of Technology), Timoteo Carletti (Univ. of Namur), Ginestra Bianconi (Queen Mary Univ.), Lorenzo Giambagli (Univ. of Florence), Lucille Calmon (INSERM Paris)
2	"A Mathematical Model and Its Mathematical Analysis Representing Filtration and Filter Clogging in an Aquarium"	Ken Furukawa (RIKEN), Hiroyuki Kitahata(Chiba Univ.)
*3	"Estimating underlying network structures from observed two-mode clustering patterns using nonlocal monostable equation with sign-changing kernel"	Ryu Fujiwara (Meiji University)
*4	"Mathematical modeling of the chemical reaction system robot which simultaneous controls sensing of the environment and movements"	Arashi Odanaka (Future University Hakodate), Yoshitaro Tanaka(Future University Hakodate), Shigeru Sakurazawa(Future University Hakodate)
5	"Segmented pattern for oscillated reaction-diffusion system"	Ayuki Sekisaka (Meiji University)
6	"Traveling Waves in a Reaction-Diffusion System for the Spread of Early Farming in Europe"	M. Humayun Kabir (Jahangirnagar University), Toshiyuki Ogawa (Meiji Univ.)
7	"Stability of Periodic Traveling Waves and Spiral wave Dynamics in Excitable Reaction-Diffusion Systems"	M. Osman Gani (Jahangirnagar University), Toshiyuki Ogawa (Meiji Univ.)
8	"Interface motion of Allen-Cahn equation with nonlinear anisotropic diffusivity"	Park Hyunjoon (Meiji University)
*9	"Blocking and propagation phenomena in spatially undulating cylindrical domains"	Ryunosuke Mori (Meiji University), Hiroshi Matano (MIMS, Meiji Univ.)
10	"Non-equilibrium Oscillatory Resonance with Delayed Dynamics"	Kenta Ohira (Nagoya University)
11	"Controlled the Eigen Frequencies by Energy Density Method"	Toshie Sasaki (Meiji University), Ichiro Hagiwara (MIMS, Meiji Univ.)

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