Mathematical Understanding of Spot Replication in Reaction-Diffusion Systems

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Phenomenon of Spot Self-replication

1. spot replication was first observed by Pearson in the Gray-Scottmodel (Science, 1993)

$$u_t = d_u \Delta u + u^2 v - au$$
$$v_t = d_v \Delta v - u^2 v + b(1-v)$$

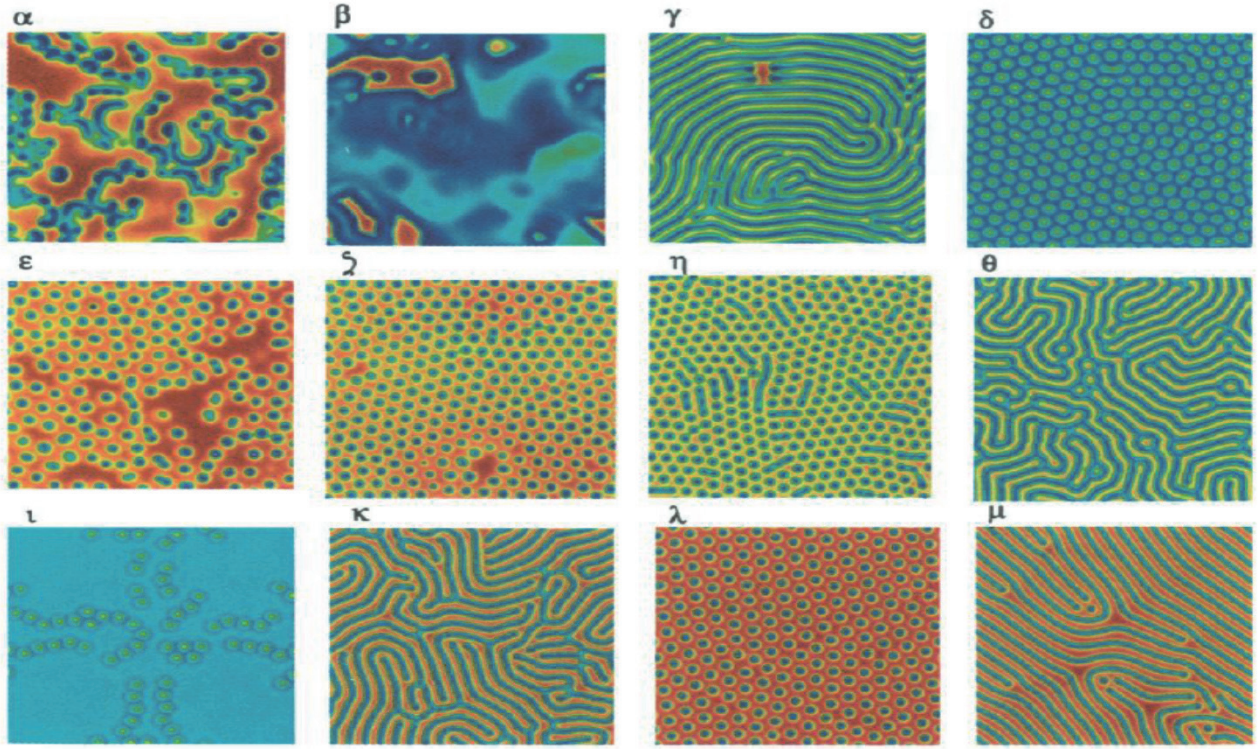


Fig. 2. The key to the map. The patterns shown in the figure are designated by Greek letters, which are used in Fig. 3 to indicate the pattern found at a given point in parameter space.

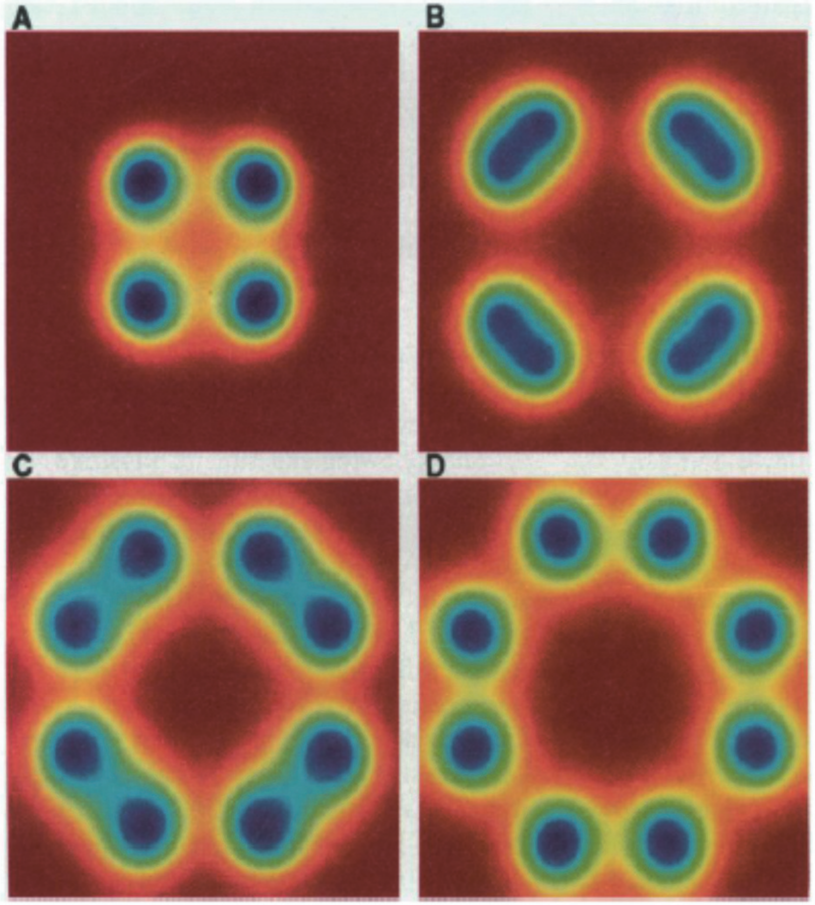


Fig. 4 (left). Time evolution of spot multiplication. This figure was produced in a 256 by 256 simulation with physical dimensions of 0.5 by 0.5 and a time step of 0.01. The times t at which the figures were taken are as follows: (**A**) $t = 0$; (**B**) $t = 350$; (**C**) $t = 510$; and (**D**) $t = 650$.

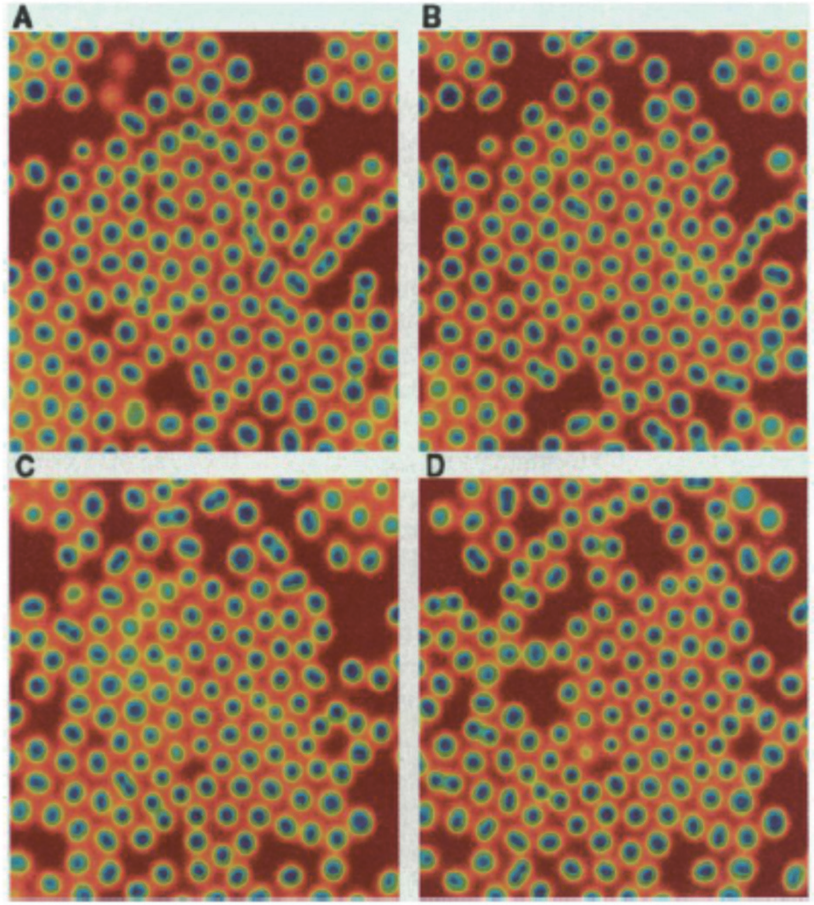


Fig. 5 (right). Time evolution of pattern ε. The images are 250 time units apart. In the corners (which map to the same point in physical space), one can see a yellow region in (**A**) to (**C**). It has decayed to red in (**D**). In (A) and (B), the center of the left border has a red region that is nearly filled in (D).

Conditions proposed by Nishiura and Ueyama

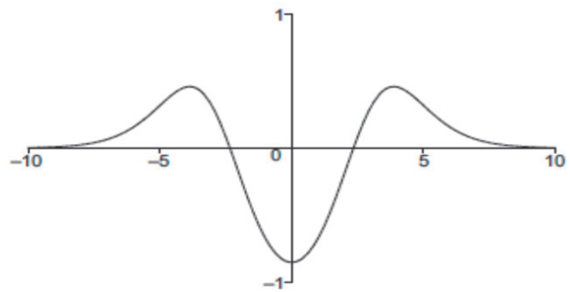
(S1) Disappearance of ground-state solution due to a fold point (saddle-node bifurcation) that occurs when a control parameter is increased above a threshold value

(S2) Existence of a dimple eigenfunction associated with a zero eigenvalue at the fold point, which is believed to be responsible for the initiation of the self-replication process

(S3) Stability of the steady-state solution on one side of the fold point near which the self-replication will occur.

(S4) Alignment of the fold points, so that the disappearance of k ground states, with k = 1, 2, 3, . . ., occurs at roughly the same value of the control parameter

Dimple shape



Simple model of spot replication

Simplified Gierer-Meinhardt model

$$u_t = \varepsilon^2 \Delta u - u + u^p v^{-l}, \quad 0 = D\Delta v - v + u^m v^{-s}$$

Shadow system $D \to \infty$, $\varepsilon \to 0$: $u_t = \Delta u - u + u^p \left(\int u^m\right)^{\frac{-l}{1+s}}$.

Consider v as a control parameter. $v^{-l} \sim \dfrac{\text{(curve)}}{\left(\int u^m\right)^{\frac{l}{1+s}}} \sim \dfrac{1 + a|x|^q}{\int u^m}$

"Simplest Model"

$$u_t = \Delta u - u + \frac{(1 + a|x|^q)\, u^p}{\int_{\mathbb{R}^N} (1 + a|x|^q) u^{p+1} dx}, \quad x \in \mathbb{R}^N;\ \nabla u(0,t) = 0, u > 0.$$

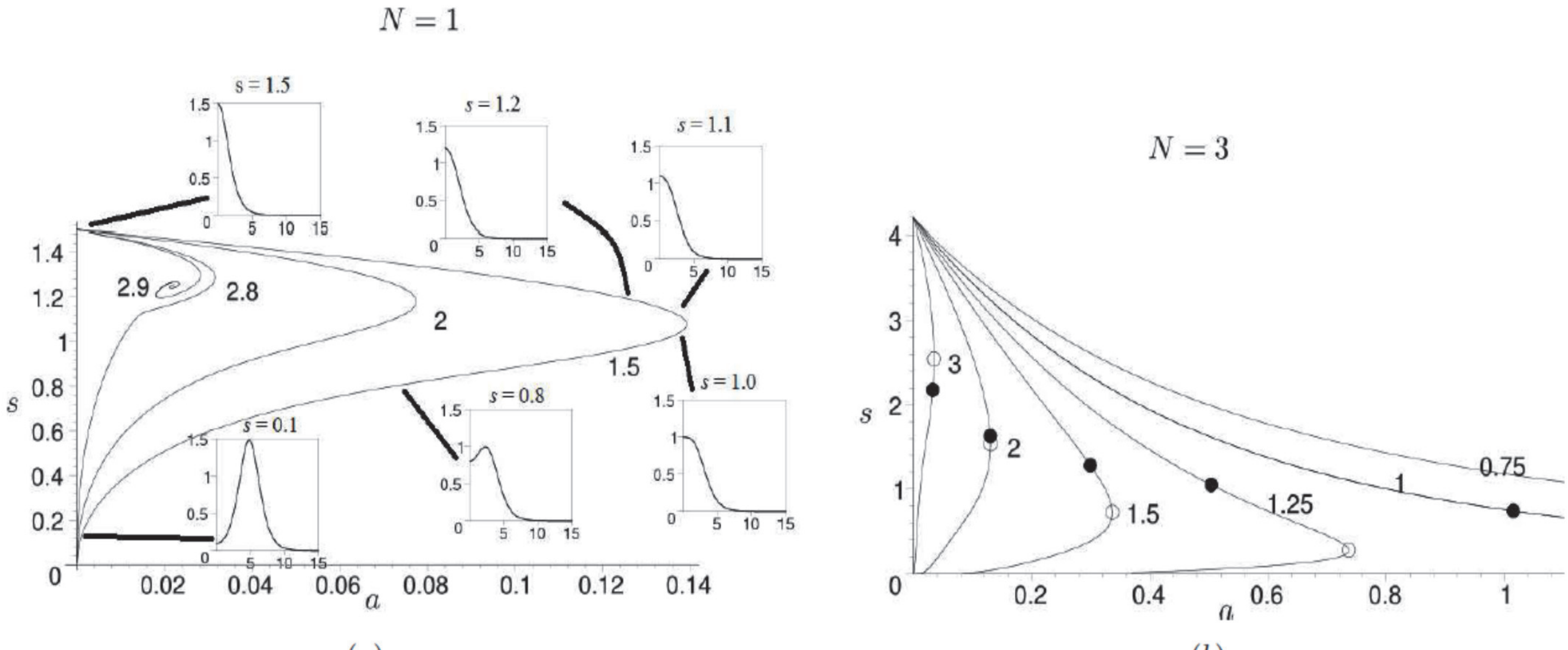
$$c_1^{1/(1-p)} u \to u, \text{ where } c_1 = \left(c_0 \int_0^\infty r^{N-1} (1 + ar^q) u^{p+1} dr\right)^{-1}$$

$$\Delta u - u + (1 + a|x|^q)\, u^p = 0,\ u > 0 \text{ in } \mathbb{R}^N$$

Bifurcation diagram

$$u_{rr} + \frac{N-1}{r} u_r - u + (1 + ar^q)\, u^p = 0,\ u'(0) = 0,\ u \to 0 \text{ as } r \to \infty.$$

$s = u(0),\ p = 2$



(a) (b)

Existence of a fold point ⟺ **Nonexistence of solutions for large “a”**

Theorem (Chen-Kolokolnikov , SIAM 2012)

$$a \geq 0,\ s = u(0),\ p^* = \begin{cases} (N+2)/(N-2), & N \geq 3 \\ \infty, & N \leq 2 \end{cases}$$

$$q^\star = \frac{(p-1)N}{2};\ q_c = \frac{(p-1)(N-1)}{2}.$$ *Then*

(i) *Suppose that* $p \in (1, p^\star)$ *and* $q \geq 0$. *Given any constant* $a_0 > 0$, *there exists a constant* $s_0 = s_0(a_0, p, q)$ *such that if* $0 \leq a < a_0$ *then the solution does not exist if* $s > s_0$.

(ii) *Suppose that either* $N \geq 3$ *and* $q > q_c$ *or else* $N \leq 2$ *and* $q > q^\star$. *There exists a constant* a_0 *such the solution does not exist if* $a > a_0$.

(iii) *If* $N \geq 3$, $0 \leq q < q_c$ *then the solution exists for all* $a \geq 0$, *provided that* $1 < p < p^\star$.

Non-local problem

Method developed by J. Wei ('99). Fold point (a_c, s_c).

Lemma

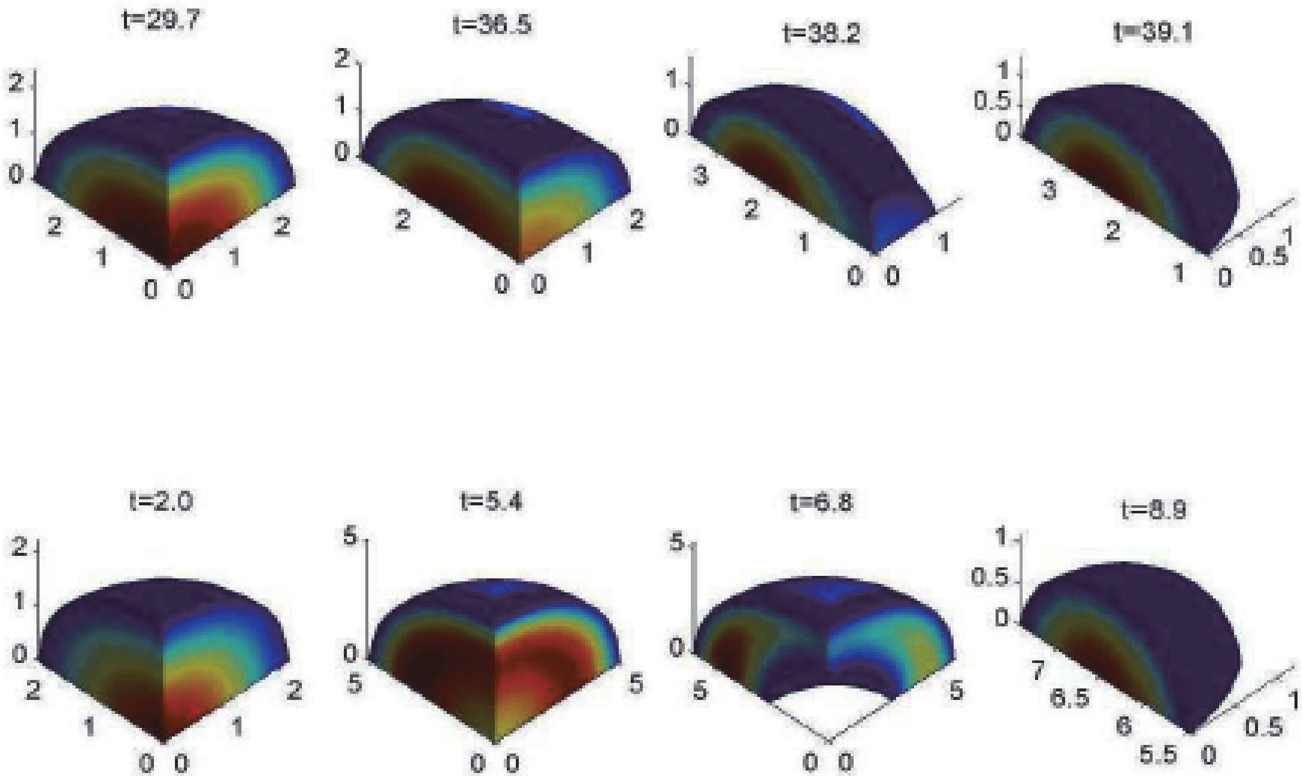
(1) Suppose the local problem admits a unique positive eigenvalue. Then the nonlocal problem is stable, i.e. it has no positive eignevalues.
(2) Suppose the local problem admits at least two positive eigenvalues. Then the nonlocal eigenvalue problem is unstable, i.e. it admits at least one positive eigenvalue.

Theorem

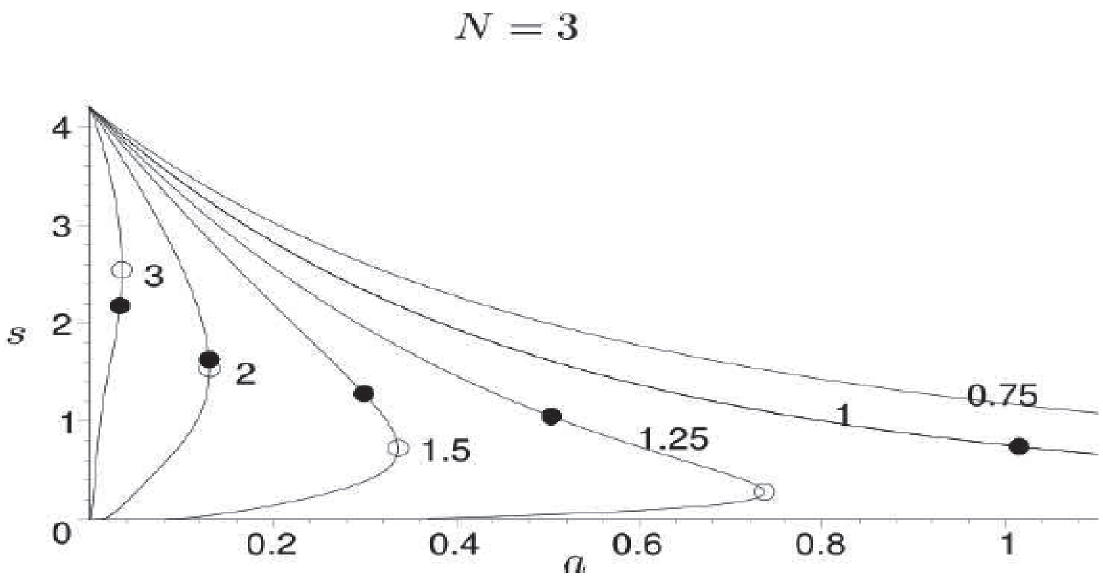
(1) If $s = s_c$, then non-local eigenvalue problem admits a zero eigenvalue whose eigenfunction is given by $Z = \frac{\partial u}{\partial s}|_{s=s_c}$. Moreover $Z(r)$ has at least one root $r > 0$. Thus Condition (S2) is proved.

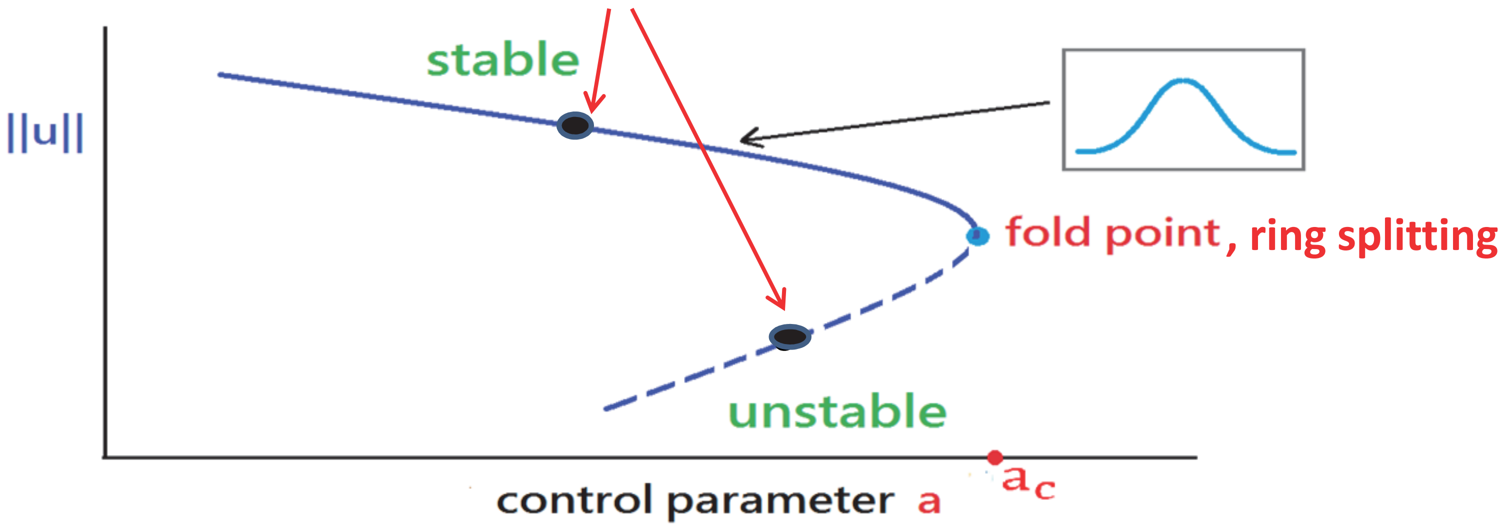
(2) It $s \in (s_c, s_0]$, then the nonlocal eigenvalue problem is stable with respect to radially symmetric perturbations. This verifies Condition (S3).

Non-radial behavior: peanut splitting



Numerical and asymptotic analysis





Conclusion

1. In general, one needs at least two equations to produce the phenomenon of spot self-replication. We propose a simple **one equation model** with a nonlocal term.
2. To obtain a fold-point structure, we prove a new **Liouville theorem** with big control parameter "**a**" for our model.
3. The conditions (S1), (S2) and (S3) proposed by Nishiura and Ueyama are **rigorously verified** for the radial case in our model.
4. The peanut splitting for the non-radial case is studied by asymptotic and numerical analysis.